DAHLGREN DIVISION
NAVAL SURFACE WARFARE CENTER
Dahlgren, Virginia 22448-5100

NSWCDD/TR-01/103

DIRAC NETWORKS:
AN APPROACH TO PROBABILISTIC INFERENCE BASED UPON THE DIRAC ALGEBRA OF QUANTUM MECHANICS

BY A. D. PARKS
   M. A. HERMANN

SYSTEMS RESEARCH AND TECHNOLOGY DEPARTMENT

JUNE 2001

20020205 080

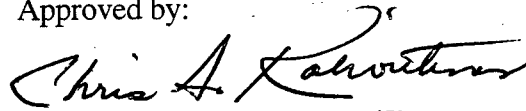# REPORT DOCUMENTATION PAGE

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, search existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE June 2001	3. REPORT TYPE AND DATES COVERED Final

4. TITLE AND SUBTITLE

Dirac Networks: An Approach to Probabilistic Inference Based Upon the Dirac Algebra of Quantum Mechanics

5. FUNDING NUMBERS

6. AUTHOR(s)

A. D. Parks and M. A. Hermann

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Commander
Naval Surface Warfare Center
Dahlgren Division (Code B35)
17320 Dahlgren Road
Dahlgren, VA 22448-5100

8. PERFORMING ORGANIZATION REPORT NUMBER

NSWCDD/TR-01/103

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

This report describes how the Dirac algebra of quantum mechanics provides for a robust and self-consistent approach to probabilistic inference system modeling and processing. We call such systems Dirac networks and demonstrate how their use: (1) allows an efficient algebraic encoding of the probabilities and distributions for all possible combinations of truth values for the logical variable in an inference system; (2) employs unitary rotation, time evolution, and translation operators to model influences upon system variable probabilities and their distributions; (3) guarantees system normalization; (4) admits unambiguously defined linear, as well as cyclic, cause and effect relationships; (5) enables the use of the von Neumann entropy as an informational uncertainty measure; and (6) allows for a variety of "measurement" operators useful for quantifying probabilistic inferences. Dirac networks should have utility in such diverse application areas as data fusion and analysis, dynamic resource allocation, qualitative analysis of complex systems, automated medical diagnostics, and interactive/collaborative decision processes. The approach is illustrated by developing and applying simple Dirac networks to the following representative problems: (a) cruise missile - target allocation decision aiding; (b) genetic disease carrier identification using ancestral evidential information; (c) combat system control methodology trade-off analysis; (d) finding rotational symmetries in a digital image; and (e) fusing observational error profiles. Optical device implementations of several Dirac network components are also briefly discussed.

14. SUBJECT TERMS

Probabilistic Inference, Dirac Networks, Dirac Algebra, Quantum Mechanics

15. NUMBER OF PAGES 106
16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORTS	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

NSN 7540-01-280-5500

i/ii

Standard Form 298 (Rev 2-89)
Prescribed by ANSI std. Z39-18
298-102

FOREWORD

This report has been prepared to document and describe a novel approach to probabilistic reasoning using the Dirac algebra of quantum mechanics. Consequently, inference systems constructed using this method cannot only be realized using traditional software/hardware engineering techniques, but can also have optical and quantum computational implementations. Such inference systems should have practical utility for combat system elements where valid decisions and reasonable resource allocations must be made in uncertain information environments.

This report has been reviewed by Dr. G. Moore, Advanced Computation Technology Division Head (B10), of the Systems Research and Technology Department (B) at the Naval Surface Warfare Center Dahlgren Division (NSWCDD).

Approved by:

CHRIS A. KALIVRETENOS, Head
Systems Research and Technology Department

CONTENTS

ILLUSTRATIONS

1 INTRODUCTION

In recent years, there has been much interest in developing practical logical systems which can be used to produce probabilistic inferences from probabilistic evidential information. The designs for such systems are generally hybrid concepts based upon the well-known theories of probability and first-order predicate logic. Some of these inference systems also include the probabilistic effects of influences between logical variables using network models. These networks are usually represented by acyclic directed graphs in which each node corresponds to a random logical variable that is associated with the application domain of interest and each arc expresses the "influence" of one variable upon another. Various schemes - which are usually based upon a Bayesian probability calculus - are used to evaluate the effects of influences and to obtain resultant probabilities for key inferences that may be logically constructed from the system's variables. Since realistic inference systems tend to include many variables and influences, maintaining a self-consistent system model can be difficult, especially in dynamic and collaborative environments where frequent changes from multiple sources are made to the system.

The *Dirac algebra* of quantum mechanics has been developed by physicists for the purpose of encoding for calculation the probabilistic nature and uncertainty characteristics intrinsic to the quantum physical world. The fact that probabilistic inference systems possess characteristics analogous to these suggests that the Dirac algebra can also be applied to inference systems for encoding and calculational purposes. This report demonstrates that the Dirac algebra does indeed provide a natural and efficient setting for the algebraic representation and processing of probabilistic inference systems. Dirac algebra based inference systems are called *Dirac networks* and show that such systems: (1) provide for a self consistent and efficient *encoding* of the probabilities and distributions for all possible combinations of truth values for the system's logical variables; (2) can utilize unitary rotation, time evolution, and translation operators to *modify* probabilities and distributions for variables that are induced by the influences of other variables and by time; (3) permit unambiguous *linear* and *controlled cyclic* cause and effect relationships through the use of operator order indices and an *ordering operator*; (4) guarantee that the system is *always* normalized; (5) enable the straightforward evaluation of a meaningful informational uncertainty measure defined in terms of the *von Neumann entropy*; and (6) allow for a variety of "measurement" operators that are useful for quantifying probabilistic inferences that may be derived from the system.

The richness and self-consistent nature of Dirac networks suggests its general applicability to such diverse areas as data fusion and analysis, dynamic resource allocation, qualitative analysis of complex systems, automated medical diagnostics, and interactive/collaborative decision processes. As is well known, the computational complexity that is associated with this class of inference problem is formidable [3], [4]. The implementation of such applications as Dirac networks using conventional software/hardware engineering techniques does not alleviate this computational complexity problem. However, the fact that the associated Dirac networks have their system specifications written in the language of quantum mechanics suggests that quantum mechanical implementations which render the computational complexity as "polynomial time"

might eventually be possible (recall that Shor's [11] quantum mechanical algorithm transforms the computational intractability of finding the prime factors of an integer on a conventional computer into a polynomial time computation on quantum mechanical computer).

This report is divided into four major sections. The first section describes how the Dirac algebra can be used to define, modify, and "measure" *truth state vectors* for probabilistic inference systems. Application of the algebraic theory to include, manipulate, and "measure" general *probability distributions* for inference system variables is discussed in the second section. The third major section is concerned with *demonstrating* the versatility of the approach by applying it to several representative classes of problems. The fourth section initiates a brief discussion concerning how certain components of Dirac networks can be implemented as optical devices.

It should be noted that this paper is intended to be as algebraically self-contained as is feasible. In keeping with this, many simple examples are used to illustrate results as they are developed. In addition, proofs to theorems are included to further aid the reader's understanding of the Dirac algebra. However, should additional background material be required, the reader is urged to consult several of the excellent contemporary textbooks [1, 2] which develop modern quantum mechanics from a Dirac algebraic perspective.

2 TRUTH STATE VECTORS AND OPERATORS FOR PROBABILISTIC INFERENCE SYSTEMS

2.1 THE TRUTH STATE FOR A LOGICAL VARIABLE

A two-dimensional *truth vector space* can be assigned to each logical variable in an inference system. The truth vector space assigned to the jth variable is spanned by the orthonormal *Dirac ket basis vectors* $\mid +\rangle_j$ and $\mid -\rangle_j$ which represent the true (+) and false (−) values that can be acquired by the jth variable (it is understood that $\pm$ means ± 1), respectively. For each such space, there also exists a dual (or adjoint) space spanned by the *Dirac bra basis vectors* $_j\langle + \mid$ and $_j\langle - \mid$. *Scalar products* can be formed from these bra-ket vectors according to the following "multiplication" rules:

$$_j\langle \pm \mid \pm\rangle_j = 1,$$

and

$$_j\langle \pm \mid \mp\rangle_j = 0.$$

These rules clearly exhibit the orthonormality property of the truth basis vectors and can be represented more compactly by the following single expression:

$$_j\langle \omega' \mid \omega\rangle_j = \delta_{\omega'\omega},$$

where $\omega, \omega' \in \{+, -\}$ and $\delta_{\omega'\omega}$ is the Kronecker delta defined by:

$$\delta_{\omega'\omega} = \begin{cases} 1 \text{ when } \omega' = \omega \\ 0 \text{ when } \omega' \neq \omega \end{cases}.$$

The general *truth state* for the jth variable is a ket vector $\mid \theta_j\rangle$ that is a linear superposition of the associated truth ket basis vectors $\mid \pm\rangle_j$. Specifically, the truth state for the jth variable is defined by

$$\mid \theta_j\rangle = \cos\theta_j \mid +\rangle_j + \sin\theta_j \mid -\rangle_j,$$

where, in order to avoid logical ambiguities, it is required that $0 \leq \theta_j \leq \frac{\pi}{2}$. Here the coefficients $\cos\theta_j$ and $\sin\theta_j$ are the *probability amplitudes* associated with the true and false values for the jth variable, respectively. The squares of these amplitudes are the probabilities of the truth values for the variable. Thus, $\cos^2\theta_j$ is the probability that the truth value for the variable is "true" and $\sin^2\theta_j$ $(= 1 - \cos^2\theta_j)$ is the probability that the truth value is "false." For example, if $\theta_j = 0$ $\left(\frac{\pi}{2}\right)$, then the associated variable is precisely true (false). More generally, suppose that the jth system variable is the assertion "it will rain tonight" with $\theta_j = \frac{\pi}{6}$. Then the truth state for this assertion is the ket $\left|\frac{\pi}{6}_j\right\rangle = \frac{\sqrt{3}}{2}\mid+\rangle_j + \frac{1}{2}\mid-\rangle_j$ so that the probability that the assertion is true is $\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$ and the probability that it is false is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

A truth state $|\theta_j\rangle$ is *normalized* if it forms a unit scalar product with its dual (adjoint) bra vector defined by

$$\langle \theta_j | = \cos\theta_{jj} \langle + | + \sin\theta_{jj} \langle - | .$$

Since scalar products are frequently used below, it is suggested that the details of the proof of the following lemma may be of pedagogical value to the reader who is unfamiliar with the mechanics of the algebraic manipulation of bra and ket vectors.

Lemma 1 *The truth state for any variable is normalized.*

Proof:

$$\langle \theta_j | \theta_j \rangle = (\cos\theta_{j\ j}\langle + | + \sin\theta_{j\ j}\langle - |)(\cos\theta_j | +\rangle_j + \sin\theta_j | -\rangle_j)$$

$$= \cos^2\theta_{j\ j}\langle + | +\rangle_j + \cos\theta_j \sin\theta_{j\ j}\langle + | -\rangle_j +$$

$$\sin\theta_j \cos\theta_{j\ j}\langle - | +\rangle_j + \sin^2\theta_{j\ j}\langle - | -\rangle_j$$

$$= \cos^2\theta_j + \sin^2\theta_j = 1.$$

Q.E.D.

This result is readily illustrated using the truth state for the above example: $\left\langle \frac{\pi}{j6} | \frac{\pi}{6_j} \right\rangle = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1.$

Bra and ket vectors admit matrix representations which are especially useful for computational purposes. In particular, the ket basis vectors $|+\rangle_j$ and $|-\rangle_j$ are represented by the column matrices

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_j \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_j,$$

respectively. The *adjoint* of a matrix (denoted by a superscript †) is the complex conjugate (denoted by a superscript *) of its transpose. Using this - along with the fact that the matrix representation for a bra vector is the adjoint of the matrix representation for the associated ket vector- yields the following row matrix representations for the bra vectors $_j\langle + |$ and $_j\langle - |$:

$$_j\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_j^\dagger$$

and

$$_j\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_j^\dagger.$$

It is easy to see from this and the definition of the truth state that $|\theta_j\rangle$ and its dual $\langle \theta_j |$ can be represented as matrices by

$$\begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix} = \cos\theta_j \begin{pmatrix} 1 \\ 0 \end{pmatrix}_j + \sin\theta_j \begin{pmatrix} 0 \\ 1 \end{pmatrix}_j$$

and

$$\left(\begin{array}{cc} \cos\theta_j & \sin\theta_j \end{array} \right) = \cos\theta_{jj} \left(\begin{array}{cc} 1 & 0 \end{array} \right) + \sin\theta_{jj} \left(\begin{array}{cc} 0 & 1 \end{array} \right),$$

respectively. We shall use the bra-ket notation to stand for both vectors and their matrix representations.

2.2 THE TRUTH OPERATOR FOR A LOGICAL VARIABLE AND ITS MEASUREMENT

A *truth operator* $\hat{\tau}_j$ can also be associated with the jth logical variable of an inference system. The ket basis vectors $\mid \pm \rangle_j$ are eigenvectors of this truth operator and they satisfy the eigenvalue equation given by

$$\hat{\tau}_j \mid \pm \rangle_j = \pm \mid \pm \rangle_j.$$

Thus, the eigenvalues $\pm$ (i.e., ± 1) can be interpreted as the result of "truth measurements" obtained from the action of $\hat{\tau}_j$ upon the associated basis eigenkets. This notion can be extended to a more generalized measure of the truth value for a variable's truth state. This quantity is the *expected truth value* for a variable's truth state and is defined for the jth variable by the product $\langle \hat{\tau}_j \rangle \equiv \langle \theta_j \mid \hat{\tau}_j \mid \theta_j \rangle$. Obviously, for the special case that $\theta_j = 0$ $(\frac{\pi}{2})$, then

$$
\begin{aligned}
\langle \hat{\tau}_j \rangle &= {}_j\langle + \mid \hat{\tau}_j \mid + \rangle_j \quad \left({}_j\langle - \mid \hat{\tau}_j \mid - \rangle_j \right) \\
&= {}_j\langle + \mid +1 \mid + \rangle_j \quad \left({}_j\langle - \mid -1 \mid - \rangle_j \right) \\
&= +1_j\langle + \mid + \rangle_j \quad \left(-1_j\langle - \mid - \rangle_j \right) \\
&= +1 \ (-1)
\end{aligned}
$$

and the expected truth value is precisely true (false). The expected truth value of a general truth state for a variable can easily be calculated using the following lemma.

Lemma 2 $\langle \hat{\tau}_j \rangle = \cos^2\theta_j - \sin^2\theta_j.$

Proof:

$$\langle \hat{\tau}_j \rangle = (\cos\theta_{jj}\langle + \mid + \sin\theta_{jj}\langle - \mid)\hat{\tau}_j(\cos\theta_j \mid + \rangle_j + \sin\theta_j \mid - \rangle_j)$$

$$
\begin{aligned}
&= \cos^2\theta_j \; {}_j\langle + \mid \hat{\tau}_j \mid + \rangle_j + \cos\theta_j \sin\theta_{jj}\langle + \mid \hat{\tau}_j \mid - \rangle_j + \\
&\quad \sin\theta_j \cos\theta_{jj} \langle - \mid \hat{\tau}_j \mid + \rangle_j + \sin^2\theta_{jj} \langle - \mid \hat{\tau}_j \mid - \rangle_j
\end{aligned}
$$

$$
\begin{aligned}
&= \cos^2\theta_{jj}\langle + \mid + \rangle_j - \cos\theta_j \sin\theta_{jj}\langle + \mid - \rangle_j + \\
&\quad \sin\theta_j \cos\theta_j \; {}_j \langle - \mid + \rangle_j - \sin^2\theta_{jj} \langle - \mid - \rangle_j \\
&= \cos^2\theta_j - \sin^2\theta_j.
\end{aligned}
$$

Q.E.D.

The expected truth value for a variable is therefore the difference between the probability

that the variable is true and the probability that it is false so that

$$-1 \leq \langle \hat{\tau}_j \rangle \leq +1.$$

Thus, when $\langle \hat{\tau}_j \rangle > 0$ (< 0), the variable is more true (false) than false (true) and when $\langle \hat{\tau}_j \rangle = 0$, it is as precisely as true as it is false. This obviously occurs when $\theta_j = \frac{\pi}{4}$ and represents a state of complete uncertainty about the truth for the associated variable. Referring to the truth state for the previous example assertion "it will rain tonight" and applying the last lemma, it is readily found that its expected truth value is $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. Hence, the truth state for this assertion represents the fact that the assertion is more true than false.

The truth operator also admits a computationally useful matrix representation defined by:

$$\tau_j = \begin{pmatrix} {}_j\langle +| \hat{\tau}_j |+\rangle_j & {}_j\langle +| \hat{\tau}_j |-\rangle_j \\ {}_j\langle -| \hat{\tau}_j |+\rangle_j & {}_j\langle -| \hat{\tau}_j |-\rangle_j \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j .$$

Observe that -as required- the truth eigenvalues are the diagonal entries of the matrix. Here, we shall adopt the general convention that the omission of the "^" symbol from any operator signifies the use of its matrix representation. Also, if such an omission occurs in an algebraic expression, then the entire expression is assumed to be in matrix form. A simple example of this which involves the truth operator is

$$\langle \theta_j| \tau_j |\theta_j \rangle \equiv \begin{pmatrix} \cos\theta_j & \sin\theta_j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix} .$$

An operator $\hat{O}$ is *Hermitean* if its matrix representation is self-adjoint, i.e. if its matrix representation is equal to the complex conjugate transpose of itself. In this case we write $\hat{O} = \hat{O}^\dagger$.

Lemma 3 *The truth operator for a logical variable is Hermitean.*

Proof:

$$\tau_j^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j^* = \tau_j.$$

Thus, $\hat{\tau}_j = \hat{\tau}_j^\dagger$. **Q.E.D.**

It is useful at this time to introduce the *identity operator* $\hat{1}_j$ for the jth logical variable. This operator is defined by its action $\hat{1}_j |\theta_j \rangle = |\theta_j \rangle$ upon a truth state and its juxtapositional property $\hat{1}_j \hat{\tau}_j = \hat{\tau}_j \hat{1}_j = \hat{\tau}_j$ with the truth operator. It is easy to see that the identity operator has the following anticipated matrix representation:

$$1_j = \begin{pmatrix} {}_j\langle +| \hat{1}_j |+\rangle_j & {}_j\langle +| \hat{1}_j |-\rangle_j \\ {}_j\langle -| \hat{1}_j |+\rangle_j & {}_j\langle -| \hat{1}_j |-\rangle_j \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_j.$$

It is also clear from this that the identity operator is Hermitean. We note that the symbol $\hat{1}$ will be used to denote a general identity operator so that, by convention, 1 is its associated matrix representation. Although "1" also denotes the unit scalar, the meaning of "1" will be clear from the context in which it is used.

The *truth uncertainty* $\Delta\tau_j$ for the jth logical variable is the quantity

$$\Delta\tau_j \equiv \sqrt{\left\langle \hat{\tau}_j^2 \right\rangle - \left\langle \hat{\tau}_j \right\rangle^2}$$

which provides a measure of the uncertainty - or "fuzziness" - associated with the expected truth such that the smaller (greater) the $\Delta\tau_j$ value, the more (less) certain the $\langle \hat{\tau}_j \rangle$ value. The truth uncertainty is easily calculated using the result of the following theorem.

Theorem 4 $\Delta\tau_j = 2\sin\theta_j \cos\theta_j$.

Proof: First observe that $\hat{\tau}_j^2 = \hat{1}_j$ (this is easily determined using the matrix representation for $\hat{\tau}_j$). Then $\left\langle \hat{\tau}_j^2 \right\rangle = 1$ so that

$$\begin{aligned} \Delta\tau_j &= \sqrt{1 - (\cos^2\theta_j - \sin^2\theta_j)^2} \\ &= \sqrt{4\sin^2\theta_j \cos^2\theta_j} \\ &= 2\sin\theta_j \cos\theta_j. \end{aligned}$$

Q.E.D.

Thus, not only is the expected truth value $\langle \hat{\tau}_j \rangle$ for variable j completely determined by θ_j, but so is $\Delta\tau_j$. Note that - as required: (i) when $\theta_j \to 0, \frac{\pi}{2}$, then $\Delta\tau_j \to 0$ and it becomes more certain that the truth value is $\langle \hat{\tau}_j \rangle$; (ii) when $\theta_j = 0, \frac{\pi}{2}$, then $\Delta\tau_j = 0$ and $\langle \hat{\tau}_j \rangle = +1, -1$ (i.e., the truth value is - with total certainty - precisely an eigenvalue), respectively; and (iii) when $\theta_j = \frac{\pi}{4}$, then $\langle \hat{\tau}_j \rangle = 0$ and the truth uncertainty has its greatest value $\Delta\tau_j = 1$.

2.3   TRUTH STATES FOR MULTI-VARIABLE SYSTEMS

The truth state vector $|\Psi\rangle$ for a system comprised of n variables is the *tensor product* of the n single variable truth states. This tensor product state is written in Dirac notation as the juxta-positions

$$|\Psi\rangle = |\theta_1\rangle |\theta_2\rangle \cdots |\theta_n\rangle$$

and corresponds to a vector in the 2^n dimensional vector space spanned by the orthonormal set of basis ket vectors which correspond to the set of all possible truth value combinations for the n variables (one tensor product basis vector represents one possible combination). Thus, $|\Psi\rangle$ is a

superposition of these basis vectors in which each vector coefficient encodes an amplitude whose square is the probability for the logical variables' truth values represented by the associated tensor product basis state.

To make this more precise, let $N = \{1, 2, \cdots, n\}$ index the variables for an inference system, $W = \{+, -\}$, and define

$$L_N = \{\omega = \omega_1 \omega_2 \cdots \omega_n \mid \omega_j \in W, j \in N\}. \tag{2-1}$$

This is the set of all possible strings of $+$'s and $-$'s of length n. Using this, the truth state vector for a system of n logical variables can be written as the superposition

$$|\Psi\rangle = \sum_{\omega \in L_N} f(\omega) |\omega\rangle. \tag{2-2}$$

Here $|\omega\rangle = |\omega_1 \omega_2 \cdots \omega_n\rangle \equiv |\omega_1\rangle |\omega_2\rangle \cdots |\omega_n\rangle$ and $f(\omega) \equiv h(\omega_1) h(\omega_2) \cdots h(\omega_n)$ is the probability amplitude for $|\omega\rangle$ with

$$h(\omega_j) \equiv \begin{cases} \cos \theta_j & \text{when } \omega_j = + \\ \sin \theta_j & \text{when } \omega_j = - \end{cases}. \tag{2-3}$$

Recall that - by definition - scalar products can only be formed between bra and ket vectors for the same logical variable, i.e. between vectors in the same space with the same subscript index. With this in mind, the following lemma shows that $\{|\omega\rangle \mid \omega \in L_N\}$ is indeed an orthonormal set of ket vectors.

Lemma 5 *If $\omega, \omega' \in L_N$, then $\langle \omega' \mid \omega \rangle = \delta_{\omega'\omega}$.*

Proof:

$$\langle \omega' \mid \omega \rangle = \langle \omega'_1 \mid \omega_1 \rangle \langle \omega'_2 \mid \omega_2 \rangle \cdots \langle \omega'_n \mid \omega_n \rangle$$
$$= \delta_{\omega'_1 \omega_1} \delta_{\omega'_2 \omega_2} \cdots \delta_{\omega'_n \omega_n} = \delta_{\omega'\omega}.$$

Q.E.D.

The matrix representation for $|\Psi\rangle$ is the usual $2^n \times 1$ tensor product column matrix in which each appropriately ordered entry corresponds to the superposition amplitude for the associated basis vector. The representation for the bra vector $\langle \Psi |$ is the analogously defined 1×2^n row matrix.

The desired normalization for $|\Psi\rangle$ is guaranteed by the next lemma.

Lemma 6 *$|\Psi\rangle$ is normalized.*

Proof: Using the facts that each variable's truth state is normalized and that scalar products are defined only for variables with identical subscripts, we have

$$\langle \Psi \mid \Psi \rangle = \langle \theta_1 \mid \theta_1 \rangle \langle \theta_2 \mid \theta_2 \rangle \cdots \langle \theta_n \mid \theta_n \rangle = \underbrace{1 \cdot 1 \cdots 1}_{n \text{ times}} = 1.$$

Q.E.D.

In order to illustrate this construction, let variable j be the above example assertion with truth state $\left|\frac{\pi}{6_j}\right\rangle$ and let variable k be the assertion "tomorrow will be sunny" with truth state $\left|\frac{\pi}{3_k}\right\rangle$. Then the tensor product truth state for this system is

$$
\begin{aligned}
|\Psi_{jk}\rangle &= \left|\frac{\pi}{6_j}\right\rangle \left|\frac{\pi}{3_k}\right\rangle \\
&= \left(\cos\frac{\pi}{6}\,|+\rangle_j + \sin\frac{\pi}{6}\,|-\rangle_j\right)\left(\cos\frac{\pi}{3}\,|+\rangle_k + \sin\frac{\pi}{3}\,|-\rangle_k\right) \\
&= \cos\frac{\pi}{6}\cos\frac{\pi}{3}\,|+\rangle_j\,|+\rangle_k + \cos\frac{\pi}{6}\sin\frac{\pi}{3}\,|+\rangle_j\,|-\rangle_k + \\
&\quad \sin\frac{\pi}{6}\cos\frac{\pi}{3}\,|-\rangle_j\,|+\rangle_k + \sin\frac{\pi}{6}\sin\frac{\pi}{3}\,|-\rangle_j\,|-\rangle_k \\
&= \frac{\sqrt{3}}{4}\,|+\rangle_j\,|+\rangle_k + \frac{3}{4}\,|+\rangle_j\,|-\rangle_k + \frac{1}{4}\,|-\rangle_j\,|+\rangle_k + \frac{\sqrt{3}}{4}\,|-\rangle_j\,|-\rangle_k \\
&= \frac{\sqrt{3}}{4}\,|++\rangle + \frac{3}{4}\,|+-\rangle + \frac{1}{4}\,|-+\rangle + \frac{\sqrt{3}}{4}\,|--\rangle,
\end{aligned}
$$

where we have simplified the notation by defining $|+\rangle_j\,|+\rangle_k \equiv |++\rangle$, etc. Observe that since $n = 2$, then $|\Psi_{jk}\rangle$ is a $2^2 = 4$ dimensional truth state vector expressed as a superposition of basis truth states given by the set

$$\mathbf{S}_{jk} = \{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$$

which corresponds to all possible combinations of "true/false" values for assertions j and k. As above, when $\omega = \omega_j\omega_k = ++$, then $h(\omega_j) = \cos\frac{\pi}{6}$, $h(\omega_k) = \cos\frac{\pi}{3}$ so that $f(\omega) = h(\omega_j)h(\omega_k) = \cos\frac{\pi}{6}\cos\frac{\pi}{3}$, etc. Choosing the basis state order given in set $\mathbf{S}_{jk}$ as the matrix order, then $|\Psi_{jk}\rangle$ can be represented by the column matrix

$$
|\Psi_{jk}\rangle = \begin{pmatrix} \frac{\sqrt{3}}{4} \\[6pt] \frac{3}{4} \\[6pt] \frac{1}{4} \\[6pt] \frac{\sqrt{3}}{4} \end{pmatrix},
$$

and similarly for the row matrix representation for $\langle_{jk}\Psi|$.

The square of the amplitude for $\mathbf{s} \in \mathbf{S}_{jk}$ is the probability for the conjunction of the truth valued assertions associated with $\mathbf{s}$. For example, $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$ is the probability for $|+-\rangle$, so that $\frac{9}{16}$ is the probability for the *conjunction* of assertion j when it is "true" with assertion k when it is "false" (i.e., "it is true that it will rain tonight" *and* "it is false that tomorrow will be sunny"). Also, notice that the sum of the squares of any subset of the amplitudes for $|\Psi_{jk}\rangle$ is

the probability for the *disjunction* of the conjunction of the associated truth valued assertions. For example, consider the assertions associated with the basis states $|+-\rangle$ and $|-+\rangle$. Then $\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{5}{8}$ is the probability that "it is true that it will rain tonight and it is false that tomorrow will be sunny" *or* "it is false that it will rain tonight and it is true that tomorrow will be sunny." Finally, since the $\mathbf{S}_{jk}$ basis vectors form an orthonormal set (i.e., $\langle lp \mid qr \rangle = 1$ when $l = q$ and $p = r$ and is 0 otherwise, where $l, p, q, r \in \{+, -\}$), it is readily verified that the last lemma holds. In particular,

$$\langle _{jk}\Psi \mid \Psi_{jk} \rangle = \left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2$$
$$= \frac{16}{16} = 1.$$

As seen from these considerations, the guaranteed normalization for a truth state vector is simply an affirmation of the fact that the "event" described by the disjunction of the conjunction of all possible truth evaluations (as defined by the associated basis vector set) for the logical variable assertions for an inference system is the *certain event* (i.e., has unit probability).

2.4 THE INFLUENCE OPERATOR FOR A LOGICAL VARIABLE AND THE ROTATION OF ITS TRUTH STATE

In a network model for an inference system with influences between variables, the influence of the jth logical variable upon the kth logical variable is represented by an arc from the jth node to the kth node of the underlying directed graph. In this case the jth node is called the *parent* node and the kth node is called the *child* node and we refer to them both as a *parent − child pair*. In a Dirac network the influence of the parent upon the child is modelled by a rotation of the truth state ket vector for the child through an angle α_{jk} defined for the pair. This rotation angle can be of *constant* value or can be *functionally dependent* upon the truth state of the parent (this functional dependence capability provides a modelling flexibility that is not typically available to conventional inference systems). Here the subscripts denote that parent j influences child k and we say that "variable j exerts an α_{jk} influence upon variable k." Note that a variable j can influence itself via an α_{jj} influence. Such an influence is called a *self influence* and appears as a directed loop on node j in the associated network model (clearly, this is a cycle). Also, if a variable is not influenced by another, then there is no such rotation and the variable is said to be *independent*.

The influence described above is achieved algebraically through application of the exponential *rotation operator* given by
$$\hat{R}_{jk}\left(\alpha_{jk}\right) \equiv e^{i\alpha_{jk}\hat{\sigma}_k}$$
to the truth state for variable k, where $\hat{\sigma}_k$ is the *influence operator* for variable k and $i = \sqrt{-1}$. The influence operator for variable k has eigenvectors $\left|\chi_\pm\right\rangle_k$ which are special linear

superpositions of the basis ket vectors $|\pm\rangle_k$. In particular,

$$\hat{\sigma}_k |\chi_\pm\rangle_k = \pm |\chi_\pm\rangle_k \,,$$

where

$$|\chi_\pm\rangle_k = \frac{\sqrt{2}}{2} [|+\rangle_k \pm i |-\rangle_k] \,.$$

The next two lemmas are easily verified from the definition of $|\chi_\pm\rangle_j$ and are stated without proof.

Lemma 7

$$_j \langle \chi_\pm | \chi_\pm \rangle_j = 1$$

and

$$_j \langle \chi_\pm | \chi_\mp \rangle_j = 0.$$

Lemma 8

$$|+\rangle_j = \frac{\sqrt{2}}{2} \left(|\chi_+\rangle_j + |\chi_-\rangle_j \right)$$

and

$$|-\rangle_j = -i\frac{\sqrt{2}}{2} \left(|\chi_+\rangle_j - |\chi_-\rangle_j \right) \,.$$

From this, it can easily be shown that $\hat{\sigma}_k$ has the following useful matrix representation in the $|\pm\rangle_k$ basis set:

$$\sigma_k \equiv \begin{pmatrix} _k\langle+| \hat{\sigma}_k |+\rangle_k & _k\langle+| \hat{\sigma}_k |-\rangle_k \\ _k\langle-| \hat{\sigma}_k |+\rangle_k & _k\langle-| \hat{\sigma}_k |-\rangle_k \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_k .$$

Lemma 9 *The influence operator for a logical variable is Hermitean.*

Proof:

$$\sigma_k^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_k^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}_k^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_k = \sigma_k.$$

Thus, $\hat{\sigma}_k = \hat{\sigma}_k^\dagger$. **Q.E.D.**

The result of the action of a rotation operator upon a child's truth state ket vector is defined by the following theorem.

Theorem 10 $\hat{R}_{jk}(\alpha_{jk}) \mid \theta_k\rangle = \mid \theta_k - \alpha_{jk}\rangle$.

Proof: First expand the operator in a power series:

$$\hat{R}_{jk}\left(\alpha_{jk}\right) = \sum_{n=0}^{\infty} \frac{\left(i\alpha_{jk}\hat{\sigma}_k\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\left(i\alpha_{jk}\hat{\sigma}_k\right)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\left(i\alpha_{jk}\hat{\sigma}_k\right)^{2n+1}}{(2n+1)!}.$$

Clearly, $i^{2n} = (-1)^n$ and $i^{2n+1} = (-1)^n i$. Also, using the above matrix representation for $\hat{\sigma}_k$, it is easily verified that $\hat{\sigma}_k^{2n} = \hat{1}_k^n = \hat{1}_k$, and $\hat{\sigma}_k^{2n+1} = \hat{1}_k\hat{\sigma}_k = \hat{\sigma}_k$, where $\hat{1}_k$ is the identity operator for variable k. Then

$$\hat{R}_{jk}\left(\alpha_{jk}\right) = \hat{1}_k \sum_{n=0}^{\infty} \frac{(-1)^n \left(\alpha_{jk}\right)^{2n}}{(2n)!} + i\hat{\sigma}_k \sum_{n=0}^{\infty} \frac{(-1)^n \left(\alpha_{jk}\right)^{2n+1}}{(2n+1)!}$$

$$= \hat{1}_k \cos\alpha_{jk} + i\hat{\sigma}_k \sin\alpha_{jk}.$$

Thus,

$$\hat{R}_{jk}\left(\alpha_{jk}\right) \mid \theta_k\rangle = \left(\hat{1}_k \cos\alpha_{jk} + i\hat{\sigma}_k \sin\alpha_{jk}\right) \mid \theta_k\rangle.$$

But,

$$(1_k \cos\alpha_{jk} + i\sigma_k \sin\alpha_{jk}) \mid \theta_k\rangle = \left[\begin{pmatrix} \cos\alpha_{jk} & 0 \\ 0 & \cos\alpha_{jk} \end{pmatrix} + \begin{pmatrix} 0 & \sin\alpha_{jk} \\ -\sin\alpha_{jk} & 0 \end{pmatrix}\right] \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha_{jk} & \sin\alpha_{jk} \\ -\sin\alpha_{jk} & \cos\alpha_{jk} \end{pmatrix} \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_k \cos\alpha_{jk} + \sin\theta_k \sin\alpha_{jk} \\ \sin\theta_k \cos\alpha_{jk} - \cos\theta_k \sin\alpha_{jk} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\theta_k - \alpha_{jk}\right) \\ \sin\left(\theta_k - \alpha_{jk}\right) \end{pmatrix} = \mid \theta_k - \alpha_{jk}\rangle.$$

Hence,

$$\hat{R}_{jk}\left(\alpha_{jk}\right) \mid \theta_k\rangle = \mid \theta_k - \alpha_{jk}\rangle.$$

Q.E.D.

Thus, the action of the rotation operator upon the truth state modifies the associated probability amplitudes by changing the values of their arguments by the angle α_{jk}. We shall refer to the initial truth state of a child node before it is rotated due to the influences exerted by any of its parents as the *prior − state* of the child. The truth state of a child after it has been rotated is called the *post − state* of the child. Clearly, the state $\mid \theta_k\rangle$ in the last theorem is the prior-state for the kth node and $\mid \theta_k - \alpha_{jk}\rangle$ is its post-state (recall from the previous section, however, that we require $0 \le \theta_k - \alpha_{jk} \le \frac{\pi}{2}$). Although this theorem generally eliminates the need to use a matrix representation for the rotation operator when processing the influences in an inference system, it is easy to see from the proof that

$$R_{jk}\left(\alpha_{jk}\right) = \begin{pmatrix} \cos\alpha_{jk} & \sin\alpha_{jk} \\ -\sin\alpha_{jk} & \cos\alpha_{jk} \end{pmatrix}.$$

2.5 COMMUTATORS, GLAUBER'S THEOREM, AND PROPERTIES OF ROTATION OPERATORS

Before further examination of the properties of rotation operators, it is first necessary to introduce the *commutator* $\left[\hat{X}, \hat{Y}\right]$ for operators $\hat{X}$ and $\hat{Y}$ defined by:

$$\left[\hat{X}, \hat{Y}\right] \equiv \hat{X}\hat{Y} - \hat{Y}\hat{X}.$$

If $\left[\hat{X}, \hat{Y}\right] = 0$, then operators $\hat{X}$ and $\hat{Y}$ are said to commute and the order in which the operators are applied to a system ket vector is irrelevant. Obviously, every operator commutes with itself. Truth and influence operators which act upon distinct ket vectors also commute. The following lemma summarizes these facts for the truth and influence operators and is stated without proof.

Lemma 11 *The truth and influence operators obey the following commutation relations ;*

$$(a)\ [\hat{\tau}_j, \hat{\tau}_k] = [\hat{\sigma}_j, \hat{\sigma}_k] = 0,$$

and

$$(b)\ [\hat{\tau}_j, \hat{\sigma}_k] = 0, j \neq k.$$

In order to evaluate the commutator $[\hat{\tau}_j, \hat{\sigma}_j]$, it is first necessary to introduce and define the *permutation operator* $\hat{\pi}_j$ for the jth logical variable. We do this in terms of its matrix representation given by

$$\pi_j \equiv \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)_j.$$

It is easy to see from this why $\hat{\pi}_j$ is called the permutation operator: when $\hat{\pi}_j$ acts upon a truth state, it interchanges the probability amplitudes associated with the truth eigenkets $|\pm\rangle$. More precisely,

$$\hat{\pi}_j |\theta_j\rangle = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)_j \left(\begin{array}{c} \cos\theta_j \\ \sin\theta_j \end{array}\right) = \left(\begin{array}{c} \sin\theta_j \\ \cos\theta_j \end{array}\right).$$

Lemma 12 $\hat{\pi}_j$ *is Hermitean.*

Proof: Obvious from the matrix representation for $\hat{\pi}_j$. **Q.E.D.**

It is now an easy matter to evaluate the commutator $[\hat{\tau}_j, \hat{\sigma}_j]$.

Lemma 13

$$[\hat{\tau}_j, \hat{\sigma}_j] = -2i\hat{\pi}_j.$$

Proof:

$$
\begin{aligned}
[\hat{\tau}_j, \hat{\sigma}_j] &\equiv \hat{\tau}_j \hat{\sigma}_j - \hat{\sigma}_j \hat{\tau}_j \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_j - \\
& \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j \\
&= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}_j - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_j \\
&= -2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_j \\
&\equiv -2i\hat{\pi}_j.
\end{aligned}
$$

Q.E.D.

The next lemma provides an interesting result that will be useful to our development below.

Lemma 14

$$
\langle \hat{\pi}_j \rangle = \Delta \tau_j.
$$

Proof:

$$
\begin{aligned}
\langle \hat{\pi}_j \rangle &= \begin{pmatrix} \cos\theta_j & \sin\theta_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_j \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix} \\
&= 2\sin\theta_j \cos\theta_j \\
&= \Delta\tau_j.
\end{aligned}
$$

Q.E.D.

Also required for our discussion of rotation operators is *Glauber's* theorem[2] which we also state without proof. This theorem defines a useful algebraic relationship for operator valued exponential functions in terms of the commutation properties of the operators.

Theorem 15 *(Glauber's Theorem)* If $\left[\hat{X}, \hat{Y} \right] = 0$, *then*

$$
e^{\hat{X}} e^{\hat{Y}} = e^{\hat{Y}} e^{\hat{X}} = e^{\hat{X}+\hat{Y}}.
$$

Now let us turn our attention to the rotation operator. An arbitrary operator $\hat{O}$ is *unitary* when $\hat{O}^\dagger \hat{O} = \hat{O}\hat{O}^\dagger = \hat{1}$. The unitarity of the rotation operator is easily proven in the following lemma.

Lemma 16 $\hat{R}_{jk}\left(\alpha_{jk} \right)$ *is unitary.*

Proof: Because $\hat{\sigma}_k$ is Hermitean, we can write

$$\hat{R}_{jk}^{\dagger}\left(\alpha_{jk}\right)\hat{R}_{jk}\left(\alpha_{jk}\right) = e^{-i\alpha_{jk}\hat{\sigma}_k^{\dagger}}e^{i\alpha_{jk}\hat{\sigma}_k} = e^{-i\alpha_{jk}\hat{\sigma}_k}e^{i\alpha_{jk}\hat{\sigma}_k}.$$

Since $[\hat{\sigma}_k, \hat{\sigma}_k] = 0$, Glauber's theorem applies so that

$$\hat{R}_{jk}^{\dagger}\left(\alpha_{jk}\right)\hat{R}_{jk}\left(\alpha_{jk}\right) = e^{i\alpha_{jk}(\hat{\sigma}_k - \hat{\sigma}_k)} = e^{\hat{0}_k} = \hat{1}_k.$$

Similarly for $\hat{R}_{jk}\left(\alpha_{jk}\right)\hat{R}_{jk}^{\dagger}\left(\alpha_{jk}\right)$. **Q.E.D.**

The unitarity of the rotation operator is important because it ensures the normalization of the post-state that results from application of a rotation operator to the associated (normalized) prior-state. This guarantee is provided by the next lemma.

Lemma 17 $\langle\theta_k \mid \hat{R}_{jk}^{\dagger}\left(\alpha_{jk}\right)\hat{R}_{jk}\left(\alpha_{jk}\right) \mid \theta_k\rangle = 1.$

Proof: Since

$$\hat{R}_{jk}\left(\alpha_{jk}\right)|\theta_k\rangle = |\theta_k - \alpha_{jk}\rangle = \cos\left(\theta_k - \alpha_{jk}\right)|+\rangle_k + \sin\left(\theta_k - \alpha_{jk}\right)|-\rangle_k,$$

then

$$\langle\theta_k - \alpha_{jk} \mid \theta_k - \alpha_{jk}\rangle = \cos^2\left(\theta_k - \alpha_{jk}\right) + \sin^2\left(\theta_k - \alpha_{jk}\right) = 1.$$

Therefore,

$$\langle\theta_k - \alpha_{jk}| \theta_k - \alpha_{jk}\rangle = \langle\theta_k \mid \hat{R}_{jk}^{\dagger}\left(\alpha_{jk}\right)\hat{R}_{jk}\left(\alpha_{jk}\right) \mid \theta_k\rangle = 1.$$

Q.E.D.

The order in which rotation operators are applied to truth states can be important when the rotation angles are functionally dependent upon the associated parent truth states. In such cases, *order indices* can be used to impose the necessary order of their application. Specifically, order indices m and n are counting numbers appearing as superscripts on rotation operators $\hat{R}_{ij}^m\left(\alpha_{ij}\right)$ and $\hat{R}_{kl}^n\left(\alpha_{kl}\right)$, respectively, which indicate that $\hat{R}_{ij}^m\left(\alpha_{ij}\right)$ must be applied to $|\theta_j\rangle$ before $\hat{R}_{kl}^n\left(\alpha_{kl}\right)$ can be applied to $|\theta_l\rangle$ when $m < n$. If $m = n$, then the order of their application is unimportant. Consequently, the following commutation relations hold when order indices are used:

Lemma 18

$$\left[\hat{R}_{ij}^m\left(\alpha_{ij}\right), \hat{R}_{kl}^n\left(\alpha_{kl}\right)\right] = 0, m = n,$$

$$\left[\hat{R}_{ij}^m\left(\alpha_{ij}\right), \hat{R}_{kl}^n\left(\alpha_{kl}\right)\right] \neq 0, m \neq n,$$

and similarly for the associated adjoint operators.

Proof: The first relation is a direct consequence of the fact that the influence operator is Hermitean and that $[\hat{\sigma}_j, \hat{\sigma}_l] = 0$. The second follows from the fact that the application order is important when order indices are different. **Q.E.D.**

Clearly, when $m = n$ for every m and n, i.e. when the order of application for any rotation operator is unimportant, the index is superfluous and should not be used (e.g., when all the rotation angles are constant). However, when order is important (e.g., when some of the rotation angles are functionally dependent upon their parent truth states), then it is required that every rotation operator used to specify a truth state be superscribed with an order index and that the operators be properly ordered. A truth state specification

$$\begin{aligned} |\Psi\rangle \;=\;\; & \hat{R}^{n_m}_{i_m j_m}\left(\alpha_{i_m j_m}\right)\hat{R}^{n_{m-1}}_{i_{m-1} j_{m-1}}\left(\alpha_{i_{m-1} j_{m-1}}\right)\cdots\hat{R}^{n_1}_{i_1 j_1}\left(\alpha_{i_1 j_1}\right)|\theta_{i_1}\rangle\cdots \\ & |\theta_{i_m}\rangle\,|\theta_{j_1}\rangle\cdots|\theta_{j_m}\rangle \end{aligned}$$

is *properly ordered* if $n_m \geq n_{m-1} \geq \cdots \geq n_1$. For the sake of simplicity the following notational convention is assumed: truth state specifications are implicitly properly ordered when order indices are omitted from the associated rotation operators. Note that this convention also subsumes all cases where rotation operator application order is unimportant. This notation is important for the formal algebraic specification of the general truth state for an inference system which possesses many influences between its logical variables.

Let J be the index set for parent variables which influence the child variable k so that $\Pi_{j\in J}\hat{R}_{jk}(\alpha_{jk})$ is the associated properly ordered string of rotation operators which act upon variable $k's$ prior truth state and $\sum_{j\in J}\alpha_{jk}$ is their cumulative rotation of the prior truth state. The following theorem gives the post truth state for a child variable that results from the cumulative influences of multiple parents.

Theorem 19 $\Pi_{j\in J}\hat{R}_{jk}\left(\alpha_{jk}\right)\mid\theta_k\rangle =\mid\theta_k - \sum_{j\in J}\alpha_{jk}\rangle$.

Proof: Since $[\hat{\sigma}_k, \hat{\sigma}_k] = 0$, Glauber's theorem applies so that

$$\Pi_{j\in J}\hat{R}_{jk}\left(\alpha_{jk}\right)\mid\theta_k\rangle = e^{i\left(\sum_{j\in J}\alpha_{jk}\right)\hat{\sigma}_k}\mid\theta_k\rangle =\mid\theta_k - \sum_{j\in J}\alpha_{jk}\rangle.$$

Q.E.D.

As mentioned above, it is required that $0 \leq \theta'_k \equiv \theta_k - \sum_{j\in J}\alpha_{jk} \leq \frac{\pi}{2}$. Thus, the following *door-stop rule* must be imposed upon truth vector post-states:

$$\theta'_k \equiv \begin{cases} 0, \text{ when } \theta_k - \sum_{j\in J}\alpha_{jk} < 0 \\ \frac{\pi}{2}, \text{ when } \theta_k - \sum_{j\in J}\alpha_{jk} > \frac{\pi}{2} \\ \theta_k - \sum_{j\in J}\alpha_{jk}, \text{ otherwise} \end{cases}.$$

It is important to note that if $|\chi_k\rangle \equiv \Pi_{j\in J}\hat{R}_{jk}\left(\alpha_{jk}\right)|\theta_k\rangle$, then

$$\langle\chi_k|\,\hat{\tau}_k\,|\chi_k\rangle = \langle\theta_k|\,\Pi_{j\in J}\hat{R}^{\dagger}_{jk}\left(\alpha_{jk}\right)\hat{\tau}_k\Pi_{j\in J}\hat{R}_{jk}\left(\alpha_{jk}\right)|\theta_k\rangle$$

is the expected truth value for the rotated truth state. Therefore, in general, $\langle\hat{\tau}_k\rangle$ can represent either the expected truth value for an independent variable or for a rotated truth state. The context in which the notation $\langle\hat{\tau}_k\rangle$ is used will make it clear whether one or the other or both of the possibilities apply.

As an illustration of these results, consider the truth state given by

$$|\Psi\rangle = \hat{R}_{jk}\left(-\frac{\pi}{18}\right)\hat{R}_{lk}\left(\frac{\pi}{9}\right)\left|\frac{\pi}{3}_j\right\rangle\left|\frac{\pi}{6}_l\right\rangle\left|\frac{\pi}{4}_k\right\rangle.$$

Note that since the rotation angles are constant, the order of application of the rotation operators is unimportant, the associated rotation operator commutator vanishes, and order indices can be omitted. Consequently, we can interchange the order of the operators in the last equation and not effect the final result. Also, the last theorem enables us to rewrite this equation as

$$\begin{aligned}
|\Psi\rangle &= \left|\frac{\pi}{3}_j\right\rangle\left|\frac{\pi}{6}_l\right\rangle e^{i(-\frac{\pi}{18}+\frac{\pi}{9})\hat{\sigma}_k}\left|\frac{\pi}{4}_k\right\rangle \\
&= \left|\frac{\pi}{3}_j\right\rangle\left|\frac{\pi}{6}_l\right\rangle\left|\frac{\pi}{4}_k - (-\frac{\pi}{18}+\frac{\pi}{9})\right\rangle \\
&= \left|\frac{\pi}{3}_j\right\rangle\left|\frac{\pi}{6}_l\right\rangle\left|\frac{7\pi}{36}_k\right\rangle.
\end{aligned}$$

Here, the influence of the jth variable is to increase the "falseness" of the kth variable, whereas that of the lth variable is to increase the "trueness" of the kth variable. The combined effect of the two influences is to generally increase the "trueness" of the kth variable's post-state. Observe that the truth value of the prior-state of the kth variable is completely uncertain because $\theta_k = \frac{\pi}{4}$. The influences induced by the jth and l^{th} variables via the rotation operators remove this uncertainty in the post-state for the kth variable. Clearly, in this case it is not necessary to employ the door-stop rule because the angle associated with the post-state truth vector lies within the required $\left[0, \frac{\pi}{2}\right]$ interval.

In order to illustrate order indices and their utility for imposing order, as well as controlling *cyclic influences* (including self influences), consider the properly ordered truth state defined by

$$\begin{aligned}
|\Psi\rangle &= \hat{R}^4_{km}\left(\alpha_{km}\right)\hat{R}^3_{lk}\left(\alpha_{lk}\right)\hat{R}^2_{kl}\left(\alpha_{kl}\right)\hat{R}^1_{jk}\left(\alpha_{jk}\right)|\theta_j\rangle|\theta_k\rangle|\theta_l\rangle|\theta_m\rangle \quad\quad (2\text{-}4) \\
&= \hat{R}^4_{km}\left(\alpha_{km}\right)\hat{R}^3_{lk}\left(\alpha_{lk}\right)\hat{R}^2_{kl}\left(\alpha_{kl}\right)|\theta_j\rangle|\theta_k - \alpha_{jk}\rangle|\theta_l\rangle|\theta_m\rangle \\
&= \hat{R}^4_{km}\left(\alpha_{km}\right)\hat{R}^3_{lk}\left(\alpha_{lk}\right)|\theta_j\rangle|\theta_k - \alpha_{jk}\rangle|\theta_l - \alpha_{kl}\rangle|\theta_m\rangle \\
&= \hat{R}^4_{km}\left(\alpha_{km}\right)|\theta_j\rangle|\theta_k - \alpha_{jk} - \alpha_{lk}\rangle|\theta_l - \alpha_{kl}\rangle|\theta_m\rangle \\
&= |\theta_j\rangle|\theta_k - \alpha_{jk} - \alpha_{lk}\rangle|\theta_l - \alpha_{kl}\rangle|\theta_m - \alpha_{km}\rangle.
\end{aligned}$$

Here variable j is independent, truth states $|\theta_k\rangle$ and $|\theta_l\rangle$ cyclically influence one another, and truth state $|\theta_k\rangle$ influences state $|\theta_m\rangle$. Although the cyclic influence is for a single cycle, it is easy to see that multiple cycles merely require the properly ordered insertion of additional appropriately indexed rotation operators in the truth state specification.

Observe that if all the rotation angles are constant for a truth state specification (and therefore independent of the associated parent truth states), then the order index is unnecessary because the same truth state is obtained regardless of the order in which the rotation operators

are applied. However - as mentioned above - this is generally not the case when rotation angles are functions of their associated parent truth states. For instance, suppose in our example that

$$\left.\begin{array}{r} \alpha_{jk} = 0.25\theta_j, \\ \alpha_{kl} = 0.10\theta_k, \\ \alpha_{lk} = \theta_l, \\ and \\ \alpha_{km} = 0.50\theta_k. \end{array}\right\} \qquad (2\text{-}5)$$

In this case, the truth state is given by

$$\begin{aligned} |\Psi\rangle \;=\; & |\theta_j\rangle\,|\theta_k - 0.25\theta_j - [\theta_l - 0.10(\theta_k - 0.25\theta_j)]\rangle \cdot \\ & |\theta_l - 0.10(\theta_k - 0.25\theta_j)\rangle \cdot \\ & |\theta_m - 0.50\{\theta_k - 0.25\theta_j - [\theta_l - 0.10(\theta_k - 0.25\theta_j)]\}\rangle\,. \end{aligned}$$

The ability to specify the order of influences via order indices is an especially important and necessary capability when rotation angles are functions of their associated parent truth states because different rotation operator application orders will generally yield different truth states. As an illustration of this, suppose the rotation operator order indices 2 and 3 are interchanged in our above specification for $|\Psi\rangle$. This produces a truth state given by

$$\begin{aligned} |\Psi\rangle \;=\; & |\theta_j\rangle\,|\theta_k - 0.25\theta_j - \theta_l\rangle\,|\theta_l - 0.10(\theta_k - 0.25\theta_j - \theta_l)\rangle \cdot \qquad (2\text{-}6) \\ & |\theta_m - 0.50(\theta_k - 0.25\theta_j - \theta_l)\rangle \end{aligned}$$

which is obviously different than that of the original specification (note however that if the rotation angles are constants, identical truth states are obtained).

Before closing this section, we introduce the *ordering operator* $\hat{\partial}_\sigma[\;]$. This operator is simply a formal mathematical convenience which can be used to explicitly denote that the string of rotation operators enclosed by its brackets is properly ordered according to some ordering scheme σ (e.g., a digraph with vertices representing variables and numbered arcs representing the order of influences). If there is only one ordering scheme under consideration, then the subscript σ is omitted. Thus $\hat{\partial}_\sigma[\;]$ is useful for simplifying the specification of a truth state (especially during its developmental phase) by: (1) enabling properly ordered strings of rotation operators to be written in any order; and (2) providing a simple notation for distinguishing between different orderings for a string of rotation operators (the number of such orderings has been studied by Fisher[5]).

2.6 THE TRUTH STATE VECTOR FOR A GENERAL INFERENCE SYSTEM

When arbitrary influences are involved in an inference system, then the associated system's truth state vector assumes the general form

$$|\,\Psi\rangle = \hat{\partial}[\mathcal{R}]\,|\,\Theta\rangle,$$

where $|\,\Theta\rangle$ is the tensor product of all the independent logical variables and prior truth state child variables in the Dirac network and $\mathcal{R}$ is the string of rotation operators, one operator for each parent-child pair in the system. We shall refer to such a state vector as a *general truth state* and the system it represents as a *general inference system*. As evidenced by the illustration in the last section, a general truth state may be rewritten as

$$|\Psi\rangle = \hat{\partial}[\mathcal{R}]\,|\Upsilon\rangle\,|\Xi\rangle$$
$$= |\Xi\rangle\,\hat{\partial}[\mathcal{R}]\,|\Upsilon\rangle,$$

where $|\,\Xi\rangle$ is the tensor product of the truth states for all the independent variables in the system and $|\,\Upsilon\rangle$ is the tensor product of the prior-states for all the child variables in the system. The following theorem guarantees the normalization of a general truth state.

Theorem 20 *Every general truth state is normalized.*

Proof: Since $|\,\Theta\rangle = |\,\Upsilon\rangle\,|\,\Xi\rangle$, then

$$|\,\Psi\rangle = \hat{\partial}[\mathcal{R}]\,|\,\Theta\rangle$$

$$= \hat{R}_{ij}\,(\alpha_{ij})\,\hat{R}_{kl}\,(\alpha_{kl})\cdots\hat{R}_{mn}\,(\alpha_{mn})\,|\,\Upsilon\rangle\,|\,\Xi\rangle.$$

Therefore,

$$\langle\Psi\,|\,\Psi\rangle = \langle\Xi\,|\,\langle\Upsilon\,|\,\hat{R}_{mn}^{\dagger}\,(\alpha_{mn})\cdots\hat{R}_{ij}^{\dagger}\,(\alpha_{ij})\,\hat{R}_{ij}\,(\alpha_{ij})\cdots\hat{R}_{mn}\,(\alpha_{mn})\,|\,\Upsilon\rangle\,|\,\Xi\rangle.$$

From the unitary and commutation properties of the rotation operators and the fact that truth states for logical variables are normalized, we obtain

$$\langle\Psi|\,\Psi\rangle = \langle\Xi\,|\,\langle\Upsilon\,|\,\hat{R}_{mn}^{\dagger}\,(\alpha_{mn})\,\hat{R}_{mn}\,(\alpha_{mn})\cdots\hat{R}_{ij}^{\dagger}\,(\alpha_{ij})\,R_{ij}\,(\alpha_{ij})\,|\,\Upsilon\rangle\,|\,\Xi\rangle$$

$$= \langle\Xi\,|\,\langle\Upsilon\,|\,\hat{1}\cdots\hat{1}\,|\,\Upsilon\rangle\,|\,\Xi\rangle$$

$$= \langle\Xi\,|\,\langle\Upsilon\,|\,\Upsilon\rangle\,|\,\Xi\rangle$$

$$= \langle\Xi\,|\,\Xi\rangle\langle\Upsilon\,|\,\Upsilon\rangle$$

$$= 1.$$

Q.E.D.

This result is easily verified using the illustration of the previous section (since the rotation angles are constant, the rotation operator ordering doesn't matter):

$$\langle\Psi|\,\Psi\rangle = \left\langle\frac{\pi}{j3}\bigg|\frac{\pi}{3j}\right\rangle\left\langle\frac{\pi}{l6}\bigg|\frac{\pi}{6l}\right\rangle\left\langle\frac{\pi}{k4}\bigg|e^{-i(-\frac{\pi}{18}+\frac{\pi}{9})\hat{\sigma}_k}e^{i(-\frac{\pi}{18}+\frac{\pi}{9})\hat{\sigma}_k}\bigg|\frac{\pi}{4}{}_k\right\rangle$$

$$= 1\cdot 1\cdot\left\langle\frac{7\pi}{k36}\bigg|\frac{7\pi}{36}{}_k\right\rangle = 1\cdot 1\cdot 1 = 1.$$

2.7   TRUTH MEASUREMENTS FOR GENERAL INFERENCE SUBSYSTEMS

The collection of truth operators for the logical variables of a general inference system may be used to construct a generalized truth operator for any subset of logical variables in the system. In particular, let N be the set of indices for the logical variables in a general inference system and let $\emptyset \neq S \subseteq N$ with cardinality $|S|$ (clearly, S defines an inference *subsystem* with vector space dimension $2^{|S|}$). Then the *generalized truth operator* for the inference subsystem with variables S is

$$\hat{T}_S \equiv \frac{1}{|S|} \sum_{j \in S} \hat{\tau}_j.$$

As the following theorem and corollary show, the expected truth value for any such subsystem is simply the average of the expected truth values for the variables in S.

Theorem 21 *If* $| \Psi \rangle$ *is the general truth state for a general inference system with a subsystem defined by the variable index set S, then*

$$\left\langle \hat{T}_S \right\rangle \equiv \langle \Psi \mid \hat{T}_S \mid \Psi \rangle = \frac{1}{|S|} \sum_{j \in S} \langle \hat{\tau}_j \rangle,$$

where $\langle \hat{\tau}_j \rangle$ *is the expected truth value for the jth logical variable.*

Proof: Let N index the variables in a general inference system and let $| \Psi \rangle = | \eta \rangle | \xi \rangle$, where $| \eta \rangle$ and $| \xi \rangle$ are the tensor products (denoted by $\oplus$)

$$| \eta \rangle = \oplus_{j \in N-S} | \mu_j \rangle \text{ and } | \xi \rangle = \oplus_{k \in S} | \phi_k \rangle.$$

Here $| \mu_j \rangle$ and $| \phi_k \rangle$ represent both independent variable states and post-states for child variables. Then

$$\langle \Psi \mid \hat{T}_S \mid \Psi \rangle = \langle \eta \mid \langle \xi \mid \hat{T}_S \mid \xi \rangle \mid \eta \rangle$$

$$= \langle \eta \mid \eta \rangle \langle \xi \mid \hat{T}_S \mid \xi \rangle$$

$$= \langle \xi \mid \hat{T}_S \mid \xi \rangle$$

$$= (\oplus_{k \in S} \langle \phi_k \mid) \left(\frac{1}{|S|} \sum_{j \in S} \hat{\tau}_j \right) (\oplus_{k \in S} \mid \phi_k \rangle)$$

$$= \frac{1}{|S|} \sum_{j \in S} \left\{ \langle \phi_j \mid \hat{\tau}_j \mid \phi_j \rangle \left(\Pi_{k \in S - \{j\}} \langle \phi_k \mid \phi_k \rangle \right) \right\}$$

$$= \frac{1}{|S|} \sum_{j \in S} \langle \phi_j \mid \hat{\tau}_j \mid \phi_j \rangle$$

$$= \frac{1}{|S|} \sum_{j \in S} \langle \hat{\tau}_j \rangle.$$

Q.E.D.

Corollary 22 $-1 \leq \langle \hat{T}_S \rangle \leq 1$.

Proof: Obviously, the maximum value for $\langle \hat{T}_S \rangle$ occurs when $\langle \hat{\tau}_j \rangle = 1$ for every $j \in S$. Then

$$\langle \hat{T}_S \rangle = \frac{1}{|S|} \sum_{j \in S} 1 = \frac{|S|}{|S|} = 1.$$

Similarly, the minimum value occurs when $\langle \hat{\tau}_j \rangle = -1$ for every $j \in S$. Then

$$\langle \hat{T}_S \rangle = \frac{1}{|S|} \sum_{j \in S} (-1) = \frac{-|S|}{|S|} = -1.$$

Q.E.D.

Analogous to the case for the expected truth value for a single variable, when $\left\langle \hat{T}_S \right\rangle > 0$ (< 0), the conjunction of the assertions of variables in S is more true (false) than false (true) and when $\left\langle \hat{T}_S \right\rangle = 0$, the conjunction is precisely as true as it is false and, thus, uncertain.

We quantify the truth uncertainty ΔT_S associated with variables in S as the average truth uncertainty given by

$$\Delta T_S = \frac{1}{|S|} \sum_{j \in S} \Delta \tau_j.$$

Lemma 23 $0 \leq \Delta T_S \leq 1$.

Proof: The maximum ΔT_S value occurs when $\Delta \tau_j = 1$ for every $j \in S$. Then

$$\Delta T_S = \frac{1}{|S|} \sum_{j \in S} 1 = \frac{|S|}{|S|} = 1.$$

When the minimum value $\Delta \tau_j = 0$ is assumed for every $j \in S$, then $\Delta T_S = 0$. **Q.E.D.**

These results can be seen in the following example where we consider the general truth state used above with $N = \{j, l, k\}$. If $S = \{l, k\}$, then $|S| = 2$ so that

$$\hat{T}_S = \left(\frac{1}{2} \right) (\hat{\tau}_l + \hat{\tau}_k)$$

and

$$
\begin{aligned}
\left\langle \hat{T}_S \right\rangle &= \left(\frac{1}{2} \right) ((\hat{\tau}_l) + \langle \hat{\tau}_k \rangle) \\
&= \left(\frac{1}{2} \right) \left\{ \left[\cos^2 \left(\frac{\pi}{6} \right) - \sin^2 \left(\frac{\pi}{6} \right) \right] + \left[\cos^2 \left(\frac{7\pi}{36} \right) - \sin^2 \left(\frac{7\pi}{36} \right) \right] \right\} \\
&= \left(\frac{1}{2} \right) \{ 0.5000 + 0.3420 \} \\
&= 0.421.
\end{aligned}
$$

Note that the post-state is used to calculate $\langle \hat{\tau}_k \rangle$. The associated truth uncertainty is

$$
\begin{aligned}
\Delta T_S &= \left(\frac{1}{2}\right)(\Delta \tau_l + \Delta \tau_k) \\
&= \left(\frac{1}{2}\right)\left[2\sin(\frac{\pi}{6})\cos(\frac{\pi}{6}) + 2\sin(\frac{7\pi}{36})\cos(\frac{7\pi}{36})\right] \\
&= \left(\frac{1}{2}\right)[0.8660 + 0.9398] \\
&= 0.9029.
\end{aligned}
$$

2.8 THE STATISTICAL OPERATOR AND VON NEUMANN ENTROPY FOR AN INFERENCE SYSTEM

As we have seen, both $\langle \hat{T}_S \rangle$ and ΔT_S measure the "truthfulness" of assertions defined by S, i.e. the truth certainty increases as $\langle \hat{T}_S \rangle \rightarrow \pm 1$ (while simultaneously $\Delta T_S \rightarrow 0$) and decreases as $\langle \hat{T}_S \rangle \rightarrow 0$ (while simultaneously $\Delta T_S \rightarrow 1$). Even though $\langle \hat{T}_S \rangle$ and ΔT_S are logic-based quantities, using them to characterize the truth uncertainty associated with Dirac networks may be adequate for many applications. However, since the theory underlying Dirac networks is physics-based, an associated entropy can be calculated to provide an alternative quantification of a system's uncertainty. Unlike $\langle \hat{T}_S \rangle$ and ΔT_S, entropy measures the *lack* of information about a general truth state. Note, however, that its value says nothing about the origin of this indeterminacy.

To define our entropy measure, it is first necessary to introduce the statistical operator. Let N and L_N be as defined above. If $\emptyset \neq A \subseteq N$ and $\omega \in L_N$, then define ϖ to be that obtained from ω by deleting from ω all ω_j such that $j \in (N - A)$ and define L_A as the set of all such ϖ. Observe that $|L_N| \geq |L_A|$ and that $\{| \varpi \rangle \mid \varpi \in L_A\}$ is a truth basis set which spans the $2^{|A|}$ dimensional vector subspace associated with the inference subsystem defined by A.

Now let

$$
| \Psi_A \rangle = \sum_{\varpi \in L_A} \nu(\varpi) | \varpi \rangle
$$

be the general truth state for the subsystem defined by A with probability amplitude $\nu(\varpi) = \nu(\omega_j \omega_k \cdots \omega_l) = h(\omega_j) h(\omega_k) \cdots h(\omega_l)$, where the function h is as defined above. The *statistical operator* $\hat{\rho}_A$ for the general inference subsystem spanned by the basis set specified by L_A is defined as

$$
\hat{\rho}_A = \sum_{\varpi \in L_A} | \varpi \rangle \nu^2(\varpi) \langle \varpi | .
$$

The *trace of* $\hat{\rho}_A$ is the sum $Tr\{\hat{\rho}_A\} \equiv \sum_{\varpi \in L_A} \langle \varpi | \hat{\rho}_A | \varpi \rangle$. As the next lemma shows, the trace of $\hat{\rho}_A$ is unity.

Lemma 24 $Tr\{\hat{\rho}_A\} = 1$.

Proof:

$$Tr\{\hat{\rho}_A\} = \sum_{\varpi \in L_A} \langle \varpi \mid \hat{\rho}_A \mid \varpi \rangle$$

$$= \sum_{\varpi \in L_A} \sum_{\varpi' \in L_A} \langle \varpi \mid \varpi' \rangle \nu^2 (\varpi') \langle \varpi' \mid \varpi \rangle$$

$$= \sum_{\varpi \in L_A} \sum_{\varpi' \in L_A} \nu^2 (\varpi') \, \delta_{\varpi' \varpi} \delta_{\varpi' \varpi}$$

$$= \sum_{\varpi \in L_A} \nu^2 (\varpi)$$

$$= \langle \Psi_A \mid \Psi_A \rangle = 1$$

since $|\Psi_A\rangle$ is normalized. **Q.E.D.**

We shall use the von Neumann entropy to provide a measure of the "informational uncertainty" associated with a general truth state. The *von Neumann entropy* E_A for a state consisting of logical variables indexed by the set A is defined as

$$E_A = -\kappa Tr\{\hat{\rho}_A \ln \hat{\rho}_A\} = -\kappa \sum_{\varpi \in L_A} \nu^2 (\varpi) \ln \nu^2 (\varpi),$$

where κ is a constant and we define $0 \cdot \ln 0 \equiv 0$. As the following results show, the von Neumann entropy has a very useful additive property which enables the straightforward calculation of its value for multi-variable systems.

Lemma 25 *If* $\mid \psi \rangle = \mid \varphi \rangle \mid \eta \rangle$*, then* $E_\psi = E_\varphi + E_\eta$.

Proof:

$$E_\psi = -\kappa Tr\left\{\hat{\rho}_\psi \ln \hat{\rho}_\psi\right\}$$

$$= -\kappa \left(Tr\left\{\hat{\rho}_\varphi \oplus \hat{\rho}_\eta \ln \hat{\rho}_\varphi + \hat{\rho}_\varphi \oplus \hat{\rho}_\eta \ln \hat{\rho}_\eta\right\}\right)$$

$$= -\kappa \left(Tr_\varphi \left\{\hat{\rho}_\varphi \ln \hat{\rho}_\varphi\right\} Tr_\eta \left\{\hat{\rho}_\eta\right\} + Tr_\varphi \left\{\hat{\rho}_\varphi\right\} Tr_\eta \left\{\hat{\rho}_\eta \ln \hat{\rho}_\eta\right\}\right)$$

$$= -\kappa \left(Tr_\varphi \left\{\hat{\rho}_\varphi \ln \hat{\rho}_\varphi\right\} + Tr_\eta \left\{\hat{\rho}_\eta \ln \hat{\rho}_\eta\right\}\right)$$

$$= E_\varphi + E_\eta,$$

where $\oplus$ denotes a tensor product and we have used the fact that the trace of a statistical operator is unity. **Q.E.D.**

Theorem 26 *Let* N *index the logical variables in a general inference system with* $\emptyset \neq A \subseteq N$. *Then,* $E_A = \sum_{j \in A} E_j$.

Proof: This is a direct consequence of the last lemma and the fact that the general truth state $\mid \Psi_A \rangle$ is a tensor product of the truth states for every variable in the index set A. **Q.E.D.**

The minimum entropy $E_{A_{\min}} = 0$ occurs when $\nu^2 (\varpi') = 1$ for some $\varpi' \in L_A$ (which implies $\nu^2 (\varpi) = 0$ for all $\varpi \in L_A - \{\varpi'\}$; recall that $0 \cdot \ln 0 = 0$), i.e., when $\mid \Psi_A \rangle = \mid \varpi' \rangle$ for some

$\varpi' \in L_A$. In this case there is complete knowledge of the truth values for the system's variables and general truth state $\mid \Psi_A \rangle$ is certain. The maximum value for E_A (i.e., the maximum informational uncertainty for $\mid \Psi_A \rangle$) occurs when it is equally probable that $\mid \Psi_A \rangle$ has the truth values given by the ket vector $\mid \varpi \rangle$ for every $\varpi \in L_A$, i.e., when $\nu(\varpi) = \frac{1}{\sqrt{|L_A|}}$ for each $\varpi \in L_A$. Then,

$$E_{A_{\max}} = -\kappa \sum_{\varpi \in L_A} \left(\frac{1}{|L_A|} \right) \ln \left(\frac{1}{|L_A|} \right)$$

$$= \kappa \ln |L_A| \, .$$

It is convenient to quantify the informational uncertainty associated with a general truth state $\mid \Psi_A \rangle$ in terms of the system *uncertainty* Δ_A defined by

$$\Delta_A \equiv \frac{E_A}{E_{A_{\max}}} = \frac{\sum_{j \in A} E_j}{\kappa \ln |L_A|} .$$

Lemma 27 $0 \leq \Delta_A \leq 1$.

Proof: Obvious. **Q.E.D.**

Thus, the closer the value of Δ_A is to 0 (1), the more certain (uncertain) the information contained in the system's general truth state $\mid \Psi_A \rangle$.

As an illustration, consider the general truth state of the previous section indexed by $N = \{j, l, k\}$. Let $A = \{j, k\}$ so that

$$\begin{aligned}
\mid \Psi_A \rangle &= \left| \frac{\pi}{3}_j \right\rangle \left| \frac{7\pi}{36}_k \right\rangle \\
&= \cos \left(\frac{\pi}{3} \right) \cos \left(\frac{7\pi}{36} \right) \mid ++ \rangle + \cos \left(\frac{\pi}{3} \right) \sin \left(\frac{7\pi}{36} \right) \mid +- \rangle + \\
&\quad \sin \left(\frac{\pi}{3} \right) \cos \left(\frac{7\pi}{36} \right) \mid -+ \rangle + \sin \left(\frac{\pi}{3} \right) \sin \left(\frac{7\pi}{36} \right) \mid -- \rangle ,
\end{aligned}$$

where - for the sake of brevity - we have suppressed the j and k subscripts in the basis state expansion. The statistical operator for this state is given by

$$\begin{aligned}
\hat{\rho}_A &= \mid ++ \rangle \cos^2 \left(\frac{\pi}{3} \right) \cos^2 \left(\frac{7\pi}{36} \right) \langle ++ \mid + \mid +- \rangle \cos^2 \left(\frac{\pi}{3} \right) \sin^2 \left(\frac{7\pi}{36} \right) \langle +- \mid + \\
&\quad \mid -+ \rangle \sin^2 \left(\frac{\pi}{3} \right) \cos^2 \left(\frac{7\pi}{36} \right) \langle -+ \mid + \mid -- \rangle \sin^2 \left(\frac{\pi}{3} \right) \sin^2 \left(\frac{7\pi}{36} \right) \langle -- \mid .
\end{aligned}$$

It is easily verified from this that

$$\begin{aligned}
Tr \{ \hat{\rho}_A \} &= \cos^2 \left(\frac{\pi}{3} \right) \cos^2 \left(\frac{7\pi}{36} \right) + \cos^2 \left(\frac{\pi}{3} \right) \sin^2 \left(\frac{7\pi}{36} \right) + \\
&\quad \sin^2 \left(\frac{\pi}{3} \right) \cos^2 \left(\frac{7\pi}{36} \right) + \sin^2 \left(\frac{\pi}{3} \right) \sin^2 \left(\frac{7\pi}{36} \right) \\
&= \cos^2 \left(\frac{\pi}{3} \right) \left[\cos^2 \left(\frac{7\pi}{36} \right) + \sin^2 \left(\frac{7\pi}{36} \right) \right] +
\end{aligned}$$

$$\sin^2\left(\frac{\pi}{3}\right)\left[\cos^2\left(\frac{7\pi}{36}\right) + \sin^2\left(\frac{7\pi}{36}\right)\right]$$
$$= \cos^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{3}\right)$$
$$= 1.$$

The von Neumann entropies for each state in the index set A are readily calculated as follows:

$$E_j = -\kappa\left\{\cos^2\left(\frac{\pi}{3}\right)\ln\left[\cos^2\left(\frac{\pi}{3}\right)\right] + \sin^2\left(\frac{\pi}{3}\right)\ln\left[\sin^2\left(\frac{\pi}{3}\right)\right]\right\} = .5624\kappa$$

and

$$E_k = -\kappa\left\{\cos^2\left(\frac{7\pi}{36}\right)\ln\left[\cos^2\left(\frac{7\pi}{36}\right)\right] + \sin^2\left(\frac{7\pi}{36}\right)\ln\left[\sin^2\left(\frac{7\pi}{36}\right)\right]\right\} = .6334\kappa.$$

Using the additive property, the entropy for $|\Psi_A\rangle$ is found to be $E_A = E_j + E_k = 1.1958\kappa$. Since $|L_A| = 2^{|A|} = 2^2 = 4$, then $E_{A_{\max}} = \kappa\ln 4 = 1.3863\kappa$ so that

$$\Delta_A = \frac{1.1958\kappa}{1.3863\kappa} = 0.8626$$

(contrast this value for Δ_A with that for ΔT_S calculated above). Notice that if there are no rotation operators associated with the state defined by the index set N, then variable k is independent and has truth state $\left|\frac{\pi}{4}{}_k\right\rangle$. The entropy for this state therefore assumes the maximum value

$$E_k = -\kappa\left\{\cos^2\left(\frac{\pi}{4}\right)\ln\left[\cos^2\left(\frac{\pi}{4}\right)\right] + \sin^2\left(\frac{\pi}{4}\right)\ln\left[\sin^2\left(\frac{\pi}{4}\right)\right]\right\} = .6931\kappa$$
$$= \kappa\ln 2 = E_{k_{\max}}.$$

In this case, the state is more informationally uncertain than that of the previous example. Specifically, $\Delta_A = 0.9056 > 0.8626$.

2.9 THE INFLUENCE ENERGY OPERATOR FOR A LOGICAL VARIABLE AND THE TIME EVOLUTION OF ITS TRUTH STATE

When the truth state for a logical variable is time dependent, then the Schrödinger equation may be used to determine an influence energy operator for the variable.

Theorem 28 *The influence energy operator for a logical variable with truth state $|\theta_j\rangle$ is given by $\hat{h}_j = \dot{\theta}_j \hat{\sigma}_j$, where $\hat{\sigma}_j$ is the influence operator and $\dot{\theta}_j = \frac{d\theta_j}{dt}$.*

Proof: The equation of motion for $|\theta_j\rangle$ is given by the Schrödinger equation

$$i\frac{d|\theta_j\rangle}{dt} = \hat{h}_j|\theta_j\rangle,$$

where $i = \sqrt{-1}$ (for obvious reasons we assume that Planck's constant has unit value and take

liberty to refer to $\hat{h}_j$ as an energy operator). Since $|\theta_j\rangle = \cos\theta_j\,|+\rangle_j + \sin\theta_j\,|-\rangle_j$, then

$$i\frac{d\,|\theta_j\rangle}{dt} = -i\dot{\theta}_j\sin\theta_j\,|+\rangle_j + i\dot{\theta}_j\cos\theta_j\,|-\rangle_j\,.$$

When this equation is expressed in matrix form, it becomes

$$
\begin{aligned}
i\frac{d}{dt}\begin{pmatrix}\cos\theta_j\\ \sin\theta_j\end{pmatrix} &= i\dot{\theta}_j\begin{pmatrix}-\sin\theta_j\\ \cos\theta_j\end{pmatrix}\\
&= i\dot{\theta}_j\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}_j\begin{pmatrix}\cos\theta_j\\ \sin\theta_j\end{pmatrix}\\
&= \dot{\theta}_j\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}_j\begin{pmatrix}\cos\theta_j\\ \sin\theta_j\end{pmatrix}.
\end{aligned}
$$

Rewriting this result in ket - operator form yields

$$i\frac{d\,|\theta_j\rangle}{dt} = \dot{\theta}_j\hat{\sigma}_j\,|\theta_j\rangle$$

from which it is readily seen that $\hat{h}_j = \dot{\theta}_j\hat{\sigma}_j$. **Q.E.D.**

Since $\hat{h}_j$ is proportional to the influence operator $\hat{\sigma}_j$ we call it the *influence energy operator* for variable j.

Lemma 29 $\hat{h}_j$ *is Hermitean.*

Proof: This is a direct consequence of the fact that $\hat{\sigma}_j$ is Hermitean and $\dot{\theta}_j$ is a real number. **Q.E.D.**

The influence energy operator for a logical variable is important because it defines a time evolution operator that describes the *influence of time* upon its associated truth state (i.e., it describes its dynamical behavior).

Theorem 30 *If a logical variable's truth state at time t_0 is $|\theta_j(t_0)\rangle$, then its truth state at time $t_1 > t_0$ is*

$$|\theta_j(t_1)\rangle = \hat{U}_j(t_1, t_0; \dot{\theta}_j)\,|\theta_j(t_0)\rangle\,,$$

where

$$\hat{U}_j(t_1, t_0; \dot{\theta}_j) = e^{-i\int_{t_0}^{t_1}\hat{h}_j\,dt} = e^{-i\hat{\sigma}_j\int_{t_0}^{t_1}\dot{\theta}_j\,dt}$$

is the time evolution operator for the jth logical variable.

Proof: Since

$$i\frac{d\,|\theta_j\rangle}{dt} = \hat{h}_j\,|\theta_j\rangle\,,$$

then

$$\int_{|\theta_j(t_0)\rangle}^{|\theta_j(t_1)\rangle}\frac{d\,|\theta_j\rangle}{|\theta_j\rangle} = -i\int_{t_0}^{t_1}\hat{h}_j\,dt$$

so that

$$\ln\left(\frac{|\theta_j(t_1)\rangle}{|\theta_j(t_0)\rangle}\right) = -i\int_{t_0}^{t_1}\hat{h}_j dt$$

or

$$|\theta_j(t_1)\rangle = e^{-i\int_{t_0}^{t_1}\hat{h}_j dt}|\theta_j(t_0)\rangle.$$

Q.E.D.

Corollary 31

$$\hat{U}_j(t_1, t_0; \dot{\theta}_j)|\theta_j(t_0)\rangle = \left|\theta_j(t_0) + \int_{t_0}^{t_1}\dot{\theta}_j dt\right\rangle.$$

Proof: Observe that since $\hat{h}_j = \dot{\theta}_j\hat{\sigma}_j$,

$$\hat{U}_j(t_1, t_0; \dot{\theta}_j) = e^{-i\hat{\sigma}_j\int_{t_0}^{t_1}\dot{\theta}_j dt}$$

has the same exponential form as a rotation operator with rotation angle $-\int_{t_0}^{t_1}\dot{\theta}dt$. Consequently, the proof is analogous to that of theorem 10. **Q.E.D.**

Lemma 32 $\hat{U}_j(t_1, t_0; \dot{\theta}_j)$ *is unitary.*

Proof: This result follows from the fact that $\hat{\sigma}_j$ is Hermitean, $[\hat{\sigma}_j, \hat{\sigma}_j] = 0$, and Glauber's theorem. **Q.E.D.**

The following lemma shows that a truth state remains normalized during its time evolution.

Lemma 33 $\langle\theta_j(t_1)|\theta_j(t_1)\rangle = \langle\theta_j(t_0)|\theta_j(t_0)\rangle = 1.$

Proof: This result follows from the unitarity of the time evolution operator. Specifically, since

$$|\theta_j(t_1)\rangle = \hat{U}_j(t_1, t_0; \dot{\theta}_j)|\theta_j(t_0)\rangle$$

then

$$\langle\theta_j(t_1)| = \langle\theta_j(t_0)|\hat{U}_j^\dagger(t_1, t_0; \dot{\theta}_j)$$

so that

$$\begin{aligned}
\langle\theta_j(t_1)|\theta_j(t_1)\rangle &= \langle\theta_j(t_0)|\hat{U}_j^\dagger(t_1, t_0; \dot{\theta}_j)\hat{U}_j(t_1, t_0; \dot{\theta}_j)|\theta_j(t_0)\rangle \\
&= \langle\theta_j(t_0)|\hat{1}|\theta_j(t_0)\rangle = \langle\theta_j(t_0)|\theta_j(t_0)\rangle = 1.
\end{aligned}$$

Q.E.D.

There is an obvious implicit natural time order that governs the order of application of multiple time evolution operators with the same truth variable subscript to the associated truth state. In particular, if $t_2 \geq t_1$, then $\hat{U}_j(t_1, t_0; \dot{\theta}_j)$ must be applied to $|\theta_j\rangle$ before $\hat{U}_j(t_3, t_2; \dot{\theta}'_j)$. When this is the case we say that $\hat{U}_j(t_3, t_2; \dot{\theta}'_j)\hat{U}_j(t_1, t_0; \dot{\theta}_j)|\theta_j(t_0)\rangle$ is a *properly time ordered evolution* (note from this that the rates of evolutionary change can be different for each time

evolution interval). Consequently, we require (and always assume) that *all time evolution spec-ifications for a variable are properly time ordered* (e.g., the previous specification is undefined if $t_0 < t_2 < t_1$).

When both time evolution operators and rotation operators are used to specify a truth state, then one of two (application dependent) cases are possible: (1) the time evolutions and rotations are independent of one another; or (2) the order in which they are applied to the truth state is important. When the former case prevails, then the truth state obtained from a time evolved child truth state is the same, regardless of the order in which the associated time evolution and rotation operators are applied to the prior-state of the child. The essence of this case is described formally in the following lemma and its corollary (however - as required above - the time evolution operators must still be properly time ordered). Clearly, the cumulative effect of rotations and time evolutions upon a truth state remain constrained by the door-stop rule.

Lemma 34 $\left[\hat{U}_j(t_1, t_0; \dot{\theta}_j), \hat{R}_{kl}^n(\alpha_{kl})\right] = 0$ *and similarly for the associated operator adjoints.*

Proof: This is a direct consequence of Glauber's theorem, the fact that the influence operator is Hermitean, and $[\hat{\sigma}_j, \hat{\sigma}_l] = 0$. **Q.E.D.**

Corollary 35 $\left[\hat{U}_j(t_1, t_0; \dot{\theta}_j), \hat{\partial}_\sigma[\mathcal{R}]\right] = 0$.

Proof: This follows directly from the last lemma and the definition of the ordering operator. **Q.E.D.**

When case (2) prevails, then order indices can be used to specify not only the application order for rotation operators, but also the application order in which time evolution operators are to be applied to a truth state relative to the application order of the rotation operators. As required, the time evolution operators must still be properly time ordered relative to themselves. The following lemma and corollary formalize this case.

Lemma 36

$$\left[\hat{U}_j^m(t_1, t_0; \dot{\theta}_j), \hat{R}_{kl}^n(\alpha_{kl})\right] = 0, m = n$$

and

$$\left[\hat{U}_j^m(t_1, t_0; \dot{\theta}_j), \hat{R}_{kl}^n(\alpha_{kl})\right] \neq 0, m \neq n.$$

Proof: The result for $m = n$ follows from Glauber's theorem and $[\hat{\sigma}_j, \hat{\sigma}_l] = 0$. The result for $m \neq n$ follows from the definition of order index. **Q.E.D.**

For notational convenience, we shall adopt the convention that $\mathcal{G}$ represents either a string of rotation operators, a string of time evolution operators, or a combined string of both rotation and time evolution operators. Furthermore, we shall also endow the ordering operator with the additional property that $\hat{\partial}_\sigma[\mathcal{G}]$ is an ordering of $\mathcal{G}$ according to a scheme σ which ensures that the time evolution operators are also properly time ordered. The next result is the obvious consequence of the unitarity of the rotation and time evolution operators and is stated without proof.

Lemma 37 $\hat{\partial}_\sigma[\mathcal{G}]$ *is unitary.*

Before closing this section, we note the following useful property for time evolution operators.

Lemma 38 *For $k \in \{1, 2, \cdots, n\}$, let $t_k \geq t_{k-1}$. Then*

$$\Pi_{k=1}^n \hat{U}_j(t_k, t_{k-1}; \dot{\theta}_j^k) \, |\theta_j\rangle = \left| \theta_k + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \dot{\theta}_j^k dt \right\rangle .$$

Proof: This is a consequence of the definition of the time evolution operator, the fact that the influence operator commutes with itself, and Glauber's theorem. **Q.E.D.**

2.10 THE INFLUENCE ENERGY STATES FOR A LOGICAL VARIABLE

The influence energy operator provides for an alternative description of a logical variable's truth state in terms of its associated influence energy basis eigenvectors instead of the usual $|\pm\rangle$ truth basis vectors. This is easily seen by recalling that the influence operator $\hat{\sigma}_j$ for a variable j has eigenvectors $|\chi_\pm\rangle_j$ which are linear superpositions of the basis ket vectors $|\pm\rangle_j$. Specifically,

$$\hat{\sigma}_j \, |\chi_\pm\rangle_j = \pm \, |\chi_\pm\rangle_j \, ,$$

where

$$|\chi_\pm\rangle_j = \frac{\sqrt{2}}{2} \left[|+\rangle_j \pm i \, |-\rangle_j \right] .$$

Since $\hat{h}_j = \dot{\theta}_j \hat{\sigma}_j$, then

$$\hat{h}_j \, |\chi_\pm\rangle_j = \pm \dot{\theta}_j \, |\chi_\pm\rangle_j \, .$$

Thus, $|\chi_\pm\rangle_j$ are the *influence energy eigenstates* for variable j and $\pm\dot{\theta}_j$ are the associated *influence energy eigenvalues*.

Lemma 8 can be used to show that the truth state for a variable can be represented in terms of its influence energy eigenstates.

Theorem 39

$$|\theta_j\rangle = \frac{\sqrt{2}}{2} \left\{ e^{-i\theta_j} \, |\chi_+\rangle_j + e^{i\theta_j} \, |\chi_-\rangle_j \right\} .$$

Proof: Substitution of the results of lemma 8 for $|\pm\rangle_j$ into $|\theta_j\rangle = \cos\theta_j \, |+\rangle_j + \sin\theta_j \, |-\rangle_j$ yields upon rearrangement

$$|\theta_j\rangle = \frac{\sqrt{2}}{2} \left\{ (\cos\theta_j - i\sin\theta_j) \, |\chi_+\rangle_j + (\cos\theta_j + i\sin\theta_j) \, |\chi_-\rangle_j \right\} .$$

Recognizing that $\cos\theta_j \pm i\sin\theta_j = e^{\pm i\theta_j}$ completes the proof. **Q.E.D.**

Note that $|\theta_j\rangle$ remains normalized in this representation. Also, observe that since

$$
\begin{aligned}
\hat{U}_j(t_k, t_{k-1}; \dot{\theta}_j^k)\, |\chi_\pm\rangle_j &= e^{-i\hat{\sigma}_j \int_{t_{k-1}}^{t_k} \dot{\theta}_j^k dt}\, |\chi_\pm\rangle_j \\
&= e^{-i(\pm 1)\int_{t_{k-1}}^{t_k} \dot{\theta}_j^k dt}\, |\chi_\pm\rangle_j
\end{aligned}
$$

(to see this, expand $\hat{U}_j(t_k, t_{k-1}; \dot{\theta}_j^k)$ in a Taylor series and use the fact that $\hat{\sigma}_j\, |\chi_\pm\rangle_j = (\pm 1)\, |\chi_\pm\rangle_j$), then

$$
\Pi_{k=1}^n \hat{U}_j(t_k, t_{k-1}; \dot{\theta}_j^k)\, |\theta_j(t_0)\rangle = \frac{\sqrt{2}}{2} \left\{ \begin{array}{l} e^{-i[\theta_j(t_0) + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \dot{\theta}_j^k dt]}\, |\chi_+\rangle_j + \\ e^{i[\theta_j(t_0) + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \dot{\theta}_j^k dt]}\, |\chi_-\rangle_j \end{array} \right\},
$$

so that the time evolution of a truth state in its influence energy representation is conveniently recorded as simple equal and opposite phase changes for each of its influence energy basis states. Of course, this state can be transformed into its $|\pm\rangle_j$ representation via the associated identity substitutions for $|\chi_\pm\rangle_j$.

2.11 THE EQUATION OF MOTION FOR THE EXPECTED TRUTH VALUE

The influence energy operator is important not only for describing the dynamics of Dirac network truth states, but also for providing equations of motion for the associated expected truth values. In this section we will derive the equation of motion for a logical variable's expected truth value and briefly discuss its significance and utility.

To derive the equation of motion for the expected truth value for the jth logical variable it is first necessary to obtain the time derivative of $\langle \hat{\tau}_j \rangle$:

$$
\begin{aligned}
\frac{d\langle \hat{\tau}_j \rangle}{dt} &= \frac{d\langle \theta_j| \hat{\tau}_j |\theta_j\rangle}{dt} \\
&= \frac{d\langle \theta_j|}{dt} \hat{\tau}_j |\theta_j\rangle + \langle \theta_j| \frac{d\hat{\tau}_j}{dt} |\theta_j\rangle + \langle \theta_j| \hat{\tau}_j \frac{d|\theta_j\rangle}{dt} \\
&= i\langle \theta_j| \hat{h}_j \hat{\tau}_j |\theta_j\rangle + \langle \theta_j| \frac{d\hat{\tau}_j}{dt} |\theta_j\rangle - i\langle \theta_j| \hat{\tau}_j \hat{h}_j |\theta_j\rangle \\
&= i\langle \theta_j| (\hat{h}_j \hat{\tau}_j - \hat{\tau}_j \hat{h}_j) |\theta_j\rangle + \langle \theta_j| \frac{d\hat{\tau}_j}{dt} |\theta_j\rangle \\
&= -i\langle \theta_j| \left[\hat{\tau}_j, \hat{h}_j\right] |\theta_j\rangle + \langle \theta_j| \frac{d\hat{\tau}_j}{dt} |\theta_j\rangle \\
&= -i\left\langle \left[\hat{\tau}_j, \hat{h}_j\right] \right\rangle + \left\langle \frac{d\hat{\tau}_j}{dt} \right\rangle,
\end{aligned}
$$

or

$$
\frac{d\langle \hat{\tau}_j \rangle}{dt} = -i\dot{\theta}_j \langle [\hat{\tau}_j, \hat{\sigma}_j] \rangle + \left\langle \frac{d\hat{\tau}_j}{dt} \right\rangle, \tag{2-7}
$$

where we have used the facts that $\hat{h}_j$ is Hermitean,

$$\frac{d\langle\theta_j|}{dt} = i\langle\theta_j|\hat{h}_j,$$

$$\frac{d|\theta_j\rangle}{dt} = -i\hat{h}_j|\theta_j\rangle,$$

and $\hat{h}_j = \dot{\theta}_j\hat{\sigma}_j$. Equation (2-7) can be further simplified by noting that $\frac{d\hat{\tau}_j}{dt} = 0$ and using lemma 13 to obtain

$$\frac{d\langle\hat{\tau}_j\rangle}{dt} = -2\dot{\theta}_j\langle\hat{\pi}_j\rangle$$

or by applying lemma 14 to rewrite this as

$$\frac{d\langle\hat{\tau}_j\rangle}{dt} = -2\dot{\theta}_j\cdot\Delta\tau_j \tag{2-8}$$

$$= -4\dot{\theta}_j\sin\theta_j\cos\theta_j.$$

Equation (2-8) is interesting because it not only relates the rate of change of the expected truth value for a variable to its truth uncertainty, but also indicates whether the state is becoming more true or more false as it changes in time. In particular, it demonstrates that:

1. if $\dot{\theta}_j = 0$, then the expected truth value for variable j is a constant of the motion;

2. for a fixed (nonzero) $\dot{\theta}_j$ the rate of change of $\langle\hat{\tau}_j\rangle$ varies directly with the value of the truth uncertainty; and

3. when $\dot{\theta}_j < 0$ (> 0) the rate is positive (negative) and the variable's truth state is moving in time towards being precisely true (false), i.e. towards the eigenstate $|+\rangle_j$ ($|-\rangle_j$).

Integration of the last equation yields the desired equation of motion. Multiplying both sides of Equation (2-8) by dt gives

$$d\langle\hat{\tau}_j\rangle = -4\cdot d\theta_j\sin\theta_j\cos\theta_j$$

which upon integration yields

$$\int_{\langle\hat{\tau}_j(t_0)\rangle}^{\langle\hat{\tau}_j(t)\rangle} d\langle\hat{\tau}_j\rangle = -4\int_{\theta_j(t_0)}^{\theta_j(t)}\sin\theta_j\cos\theta_j d\theta_j$$

or

$$\langle\hat{\tau}_j(t)\rangle = \langle\hat{\tau}_j(t_0)\rangle - 2\left\{\sin^2\theta_j(t) - \sin^2\theta_j(t_0)\right\}, \tag{2-9}$$

where

$$\theta_j(t) = \theta_j(t_0) + \int_{t_0}^{t}\dot{\theta}_j dt \tag{2-10}$$

(clearly, the door-stop rules apply to $\theta_j(t)$). Equation (2-9) is *the equation of motion for the expected truth value for a variable*. It is interesting to note that the term enclosed in curly braces in Equation (2-9) is the difference in the probabilities at times t and t_0 that the assertion

associated with variable j is false. Observe that Equation (2-9) can also be rewritten as

$$\langle \hat{\tau}_j(t) \rangle = \langle \hat{\tau}_j(t_0) \rangle + 2\{\cos^2 \theta_j(t) - \cos^2 \theta_j(t_0)\}$$

in which the term enclosed in curly braces is the difference in the probabilities at times t and t_0 that the assertion associated with variable j is true. Consequently, the equation of motion for $\langle \hat{\tau}_j \rangle$ is (as might be expected) directly related to the temporal changes in the probability that variable j is true or is false.

Thus - when given $\dot{\theta}_j$ - if the expected truth value for variable j and θ_j are known at time t_0, then Equation (2-9) can be used to predict the expected truth value for variable j at time t. As a simple illustration of this, assume that: (i) $\dot{\theta}_j$ has a constant value of -1.5^0 per hour; and (ii) $\theta_j(t_0) = 30^0$. Then from Equation (2-10) we have

$$
\begin{aligned}
\theta_j(t) &= 30^0 + (-1.5^0/hr.) \int_{t_0}^{t} dt \\
&= 30 - 1.5 \cdot (t - t_0) \ \deg.
\end{aligned}
$$

Since

$$\langle \hat{\tau}_j(t_0) \rangle = \cos^2 30^0 - \sin^2 30^0 = 0.50,$$

then - using Equation (2-9) - we have for the associated equation of motion the (time dependent) expression given by

$$
\begin{aligned}
\langle \hat{\tau}_j(t) \rangle &= 0.50 - 2\left\{\sin^2[30 - 1.5 \cdot (t - t_0)] - 0.25\right\} \\
&= 1.00 - 2\sin^2[30 - 1.5 \cdot (t - t_0)].
\end{aligned}
$$

For example, if $t - t_0 = 5hrs$, then $\langle \hat{\tau}_j(t_0 + 5) \rangle = 0.7071$.

Before concluding this section, we point out that the *equation of motion for the expected truth value* $\langle \hat{T}_S \rangle$ is (essentially) the averaged sum of the equations of motion for each variable in the set S. This is easily seen since

$$\frac{d\langle \hat{T}_S \rangle}{dt} = \frac{1}{|S|} \left\{ \sum_{j \in S} \frac{d\langle \hat{\tau}_j \rangle}{dt} \right\}$$

which upon integration yields

$$
\begin{aligned}
\langle \hat{T}_S(t) \rangle &= \langle \hat{T}_S(t_0) \rangle + \frac{1}{|S|} \sum_{j \in S} \int_{t_0}^{t} \left(\frac{d\langle \hat{\tau}_j \rangle}{dt} \right) dt \\
&= \langle \hat{T}_S(t_0) \rangle + \frac{1}{|S|} \sum_{j \in S} \{\langle \hat{\tau}_j(t) \rangle - \langle \hat{\tau}_j(t_0) \rangle\} \\
&= \langle \hat{T}_S(t_0) \rangle - \frac{2}{|S|} \sum_{j \in S} \left\{ \sin^2 \theta_j(t) - \sin^2 \theta_j(t_0) \right\}.
\end{aligned}
$$

2.12 THE TRUTH AUTOCORRELATION FUNCTION AND CONDITIONAL PROBABILITY

The correlation between two truth states $|\theta_j\rangle$ and $\left|\theta_j'\right\rangle$ for a logical variable is given by the value of the associated *truth autocorrelation function*

$$\left\langle\theta_j'\middle|\,\theta_j\right\rangle = \cos(\theta_j' - \theta_j).$$

(This expression is the direct consequence of the orthonormality of the truth basis ket vectors). Since $0 \leq \theta_j, \theta_j' \leq \frac{\pi}{2}$, then $0 \leq \left\langle\theta_j'\middle|\,\theta_j\right\rangle \leq 1$. Thus, the closer the function's value is to 1 (0) the more (less) similar the states are and the better (worse) their correlation. When the value of the function is 1, i.e. $\theta_j' = \theta_j$, (is 0, i.e. $\theta_j' - \theta_j) = \pm\frac{\pi}{2}$) the truth states are exactly correlated (are completely uncorrelated). Note that due to the constraint imposed by the door-stop rule, this latter case of complete uncorrelation can occur only when $\left|\theta_j'\right\rangle = \left|\frac{\pi}{2}_j\right\rangle = |-\rangle_j$ and $|\theta_j\rangle = |0_j\rangle = |+\rangle_j$, or vice versa.

The truth autocorrelation function can be especially useful for analyzing correlations between the truth states for a variable in the following cases:

1. the correlation between a variable's prior-state and post-state, i.e.

$$\text{if } \left|\theta_j'\right\rangle = \hat{\partial}_\sigma[\mathcal{R}]\,|\theta_j\rangle\,, \text{then } \left\langle\theta_j'\middle|\,\theta_j\right\rangle = \langle\theta_j|\left(\hat{\partial}_\sigma[\mathcal{R}]\right)^\dagger|\theta_j\rangle\,;$$

2. the correlation between a variable's time evolving truth state at different times, i.e.

$$\text{if } |\theta_j(t_1)\rangle = \hat{U}_j(t_1, t_0; \dot{\theta}_j)\,|\theta_j(t_0)\rangle\,, \text{then}$$
$$\langle\theta_j(t_1)|\,\theta_j(t_0)\rangle = \langle\theta_j(t_0)|\,\hat{U}_j^\dagger(t_1, t_0; \dot{\theta}_j)\,|\theta_j(t_0)\rangle\,;$$

   and

3. the correlation between a variable's truth state and its resultant state after a combination of rotations and time evolutions, i.e.

$$\text{if } \left|\theta_j'\right\rangle = \hat{\partial}_\sigma[\mathcal{G}]\,|\theta_j\rangle\,, \text{ then } \left\langle\theta_j'\middle|\,\theta_j\right\rangle = \langle\theta_j|\left(\hat{\partial}_\sigma[\mathcal{G}]\right)^\dagger|\theta_j\rangle\,.$$

This notion can be extended to define the truth autocorrelation function for two general inference systems

$$|\Psi\rangle = \hat{\partial}_\sigma[\mathcal{G}]\,|\Theta\rangle$$

and

$$|\Psi'\rangle = \hat{\partial}_\lambda[\mathcal{G}']\,|\Theta\rangle$$

to be

$$\langle\Psi'|\,\Psi\rangle = \langle\Theta|\left(\hat{\partial}_\lambda[\mathcal{G}']\right)^\dagger\left(\hat{\partial}_\sigma[\mathcal{G}]\right)|\Theta\rangle\,.$$

Before closing, we note that the square (modulus) of the truth autocorrelation function

$$\left| \left\langle \theta'_j \middle| \theta_j \right\rangle \right|^2 = \cos^2(\theta'_j - \theta_j)$$

may be interpreted as the *conditional probability* $p(\theta'_j \mid \theta_j)$, i.e. the probability that variable j is in truth state $\left| \theta'_j \right\rangle$ given that it was in truth state $|\theta_j\rangle$. Similarly, the square (modulus) $|\langle \Psi' | \Psi \rangle|^2$ may be interpreted as the conditional probability $p(\Psi' \mid \Psi)$ that the inference system is in truth state $|\Psi'\rangle$ given that it was in truth state $|\Psi\rangle$.

3 TRUTH DISTRIBUTION FUNCTIONS AND OPERATORS FOR PROBABILISTIC INFERENCE SYSTEMS

The formalism developed in this section is an additional and natural feature that is available for use by Dirac networks and is not generally available to traditional inference systems. It can be especially valuable for physical (e.g. optical) ·implementations of Dirac networks in which actual measurements are made upon the system.

3.1 THE TRUTH DISTRIBUTION FUNCTION FOR A LOGICAL VARIABLE

In addition to the two dimensional truth vector space, an uncountable set of position ket (and bra) basis vectors $\left\{ \left| \varphi_j \right\rangle \right\}$ $\left(\text{and } \left\{ \left\langle \varphi_j \right| \right\} \right)$ can be associated with the jth logical variable in a general inference system. The scalar product for position vectors for variable j is given by

$$\left\langle \varphi_j \middle| \varphi_j' \right\rangle = \delta \left(\varphi_j - \varphi_j' \right).$$

Here, $\delta \left(\varphi_j - \varphi_j' \right)$ is the Dirac delta function defined by

$$\delta \left(\varphi_j - \varphi_j' \right) = \left\{ \begin{array}{l} 1 \text{ when } \varphi_j = \varphi_j' \\ 0 \text{ when } \varphi_j \neq \varphi_j' \end{array} \right.$$

and has the property that for any function $f \left(\varphi_j \right)$

$$\int f \left(\varphi_j \right) \delta \left(\varphi_j - \varphi_j' \right) d\varphi_j = f \left(\varphi_j' \right).$$

Unless otherwise stated, all upper and lower integration limits are assumed to be $+\infty$ and $-\infty$, respectively.

A normalized position probability amplitude ket vector $\left| \psi_j \right\rangle$ (as well as its dual bra vector $\left\langle \psi_j \right|$) can also be assigned to the jth variable and used to represent the probability amplitude associated with some property v of its truth state (e.g. $\hat{\tau}_j$.). The ket vector $\left| \psi_j \right\rangle$ can be *projected* into the position space of the variable by forming the scalar product $\left\langle \varphi_j \middle| \psi_j \right\rangle$. This product is called the *wave function* for the jth variable's truth state and $\left| \left\langle \varphi_j \middle| \psi_j \right\rangle \right|^2$ is the associated probability distribution function for that truth state, i.e. the *truth distribution function*. Thus, $\left| \left\langle \varphi_j \middle| \psi_j \right\rangle \right|^2 d\varphi_j$ represents the probability for finding the property v for the truth state $\left| \theta_j \right\rangle$ in the interval $d\varphi_j$.

The *closure operator* for the jth variable is defined as

$$\int \left|\varphi_j\right\rangle \left\langle\varphi_j\right| d\varphi_j \equiv \hat{1}_j.$$

As the notation suggests, this operator is an identity operator comprised of projections into both the position ket and bra vector spaces. It is especially useful for producing probability distribution functions (and moments) when applied to scalar products of position amplitude vectors. This utility is demonstrated in the proof for the next theorem which provides a normalization guarantee for the truth distribution function.

Theorem 40 *If $\left|\psi_j\right\rangle$ is normalized, then so is $\left|\left\langle\varphi_j\right| \psi_j\right\rangle\right|^2$.*

Proof:

$$
\begin{aligned}
1 &= \left\langle\psi_j\middle|\psi_j\right\rangle = \left\langle\psi_j\middle|\hat{1}_j\middle|\psi_j\right\rangle = \int \left\langle\psi_j\middle|\varphi_j\right\rangle \left\langle\varphi_j\middle|\psi_j\right\rangle d\varphi_j \\
&= \int \left\langle\varphi_j\middle|\psi_j\right\rangle^* \left\langle\varphi_j\middle|\psi_j\right\rangle d\varphi_j = \int \left|\left\langle\varphi_j\middle|\psi_j\right\rangle\right|^2 d\varphi_j.
\end{aligned}
$$

Q.E.D.

We also define two operators useful for describing the properties of truth distribution functions. The first of these is the (necessarily Hermitean) *position operator* $\hat{\varphi}_j$ for the jth variable. The action of $\hat{\varphi}_j$ upon position vectors provides a formalism for position "measurements" via the eigenvalue equation

$$\hat{\varphi}_j \left|\varphi_j\right\rangle = \varphi_j \left|\varphi_j\right\rangle.$$

(Since we are not concerned with matrix representations for $\hat{\varphi}_j$, there is no confusion concerning the meanings of $\hat{\varphi}_j$ and φ_j). The *expected position* for the jth variable $\left\langle\hat{\varphi}_j\right\rangle$ is defined as

$$\left\langle\hat{\varphi}_j\right\rangle \equiv \left\langle\psi_j\middle|\hat{\varphi}_j\middle|\psi_j\right\rangle.$$

The next lemma will be of value to our discussion of the properties of the truth distribution function.

Lemma 41 *(The Moments Lemma)* *For each positive integer n,*

$$\left\langle\hat{\varphi}_j^n\right\rangle \equiv \left\langle\psi_j\middle|\hat{\varphi}_j^n\middle|\psi_j\right\rangle = \int \varphi_j^n \left|\left\langle\varphi_j\middle|\psi_j\right\rangle\right|^2 d\varphi_j.$$

Proof:

$$
\begin{aligned}
\left\langle\psi_j\middle|\hat{\varphi}_j^n\middle|\psi_j\right\rangle &= \left\langle\psi_j\middle|\hat{1}_j\hat{\varphi}_j^n\hat{1}_j\middle|\psi_j\right\rangle \\
&= \int\int \left\langle\psi_j\middle|\varphi_j'\right\rangle \left\langle\varphi_j'\middle|\hat{\varphi}_j^n\middle|\varphi_j\right\rangle \left\langle\varphi_j\middle|\psi_j\right\rangle d\varphi_j' d\varphi_j \\
&= \int\int \varphi_j^n \left\langle\psi_j\middle|\varphi_j'\right\rangle \left\langle\varphi_j'\middle|\varphi_j\right\rangle \left\langle\varphi_j\middle|\psi_j\right\rangle d\varphi_j' d\varphi_j
\end{aligned}
$$

$$= \int \int \varphi_j^n \left\langle \psi_j \middle| \varphi_j' \right\rangle \delta \left(\varphi_j' - \varphi_j \right) \left\langle \varphi_j \middle| \psi_j \right\rangle d\varphi_j' d\varphi_j$$

$$= \int \varphi_j^n \left\langle \psi_j \middle| \varphi_j \right\rangle \left\langle \varphi_j \middle| \psi_j \right\rangle d\varphi_j$$

$$= \int \varphi_j^n \left\langle \varphi_j \middle| \psi_j \right\rangle^* \left\langle \varphi_j \middle| \psi_j \right\rangle d\varphi_j$$

$$= \int \varphi_j^n \left| \left\langle \varphi_j \middle| \psi_j \right\rangle \right|^2 d\varphi_j.$$

Q.E.D.

The second operator of interest is the *positional uncertainty operator* $\hat{\Delta}\varphi_j$ for the jth variable's truth state and it is defined as

$$\hat{\Delta}\varphi_j \equiv \hat{\varphi}_j - \left\langle \hat{\varphi}_j \right\rangle.$$

As the next theorem shows, this operator is useful for providing a measure of the dispersion that is associated with a variable's truth distribution function.

Theorem 42 *If* $\left| \left\langle \varphi_j \middle| \psi_j \right\rangle \right|^2$ *is a normalized truth distribution function with* ν *as its mean, then*

$$\left\langle \hat{\varphi}_j \right\rangle = \nu$$

and $\sqrt{\left\langle \psi_j \middle| \left(\hat{\Delta}\varphi_j \right)^2 \middle| \psi_j \right\rangle}$ *is its standard deviation.*

Proof: If $\left| \left\langle \varphi_j \middle| \psi_j \right\rangle \right|^2$ is a normalized truth distribution function with mean ν, then $\left\langle \hat{\varphi}_j \right\rangle = \nu$ follows directly from the Moments Lemma when $n = 1$. Also,

$$\left\langle \psi_j \middle| \left(\hat{\Delta}\hat{\varphi}_j \right)^2 \middle| \psi_j \right\rangle = \left\langle \psi_j \middle| \left(\hat{\varphi}_j - \left\langle \hat{\varphi}_j \right\rangle \right)^2 \middle| \psi_j \right\rangle$$

$$= \left\langle \psi_j \middle| \hat{1}_j \left(\hat{\varphi}_j - \left\langle \hat{\varphi}_j \right\rangle \right)^2 \hat{1}_j \middle| \psi_j \right\rangle$$

$$= \int \int \left\langle \psi_j \middle| \varphi_j' \right\rangle \left\langle \varphi_j' \middle| \left(\hat{\varphi}_j - \left\langle \hat{\varphi}_j \right\rangle \right)^2 \middle| \varphi_j \right\rangle \left\langle \varphi_j \middle| \psi_j \right\rangle d\varphi_j' d\varphi_j$$

$$= \int \int \left\langle \psi_j \middle| \varphi_j' \right\rangle \left\langle \varphi_j' \middle| \left(\begin{array}{c} \hat{\varphi}_j^2 - \\ 2\left\langle \hat{\varphi}_j \right\rangle \hat{\varphi}_j + \\ \left\langle \hat{\varphi}_j \right\rangle^2 \end{array} \right) \middle| \varphi_j \right\rangle \left\langle \varphi_j \middle| \psi_j \right\rangle d\varphi_j' d\varphi_j$$

$$= \int \int \left\langle \psi_j \middle| \varphi_j' \right\rangle \left(\begin{array}{c} \left\langle \varphi_j' \middle| \hat{\varphi}_j^2 \middle| \varphi_j \right\rangle - \\ 2\left\langle \hat{\varphi}_j \right\rangle \left\langle \varphi_j' \middle| \hat{\varphi}_j \middle| \varphi_j \right\rangle + \\ \left\langle \hat{\varphi}_j \right\rangle^2 \left\langle \varphi_j' \middle| \varphi_j \right\rangle \end{array} \right) \left\langle \varphi_j \middle| \psi_j \right\rangle d\varphi_j' d\varphi_j$$

$$= \int \int \left(\begin{array}{c} \varphi_j^2 - \\ 2\left\langle \hat{\varphi}_j \right\rangle \varphi_j + \\ \left\langle \hat{\varphi}_j \right\rangle^2 \end{array} \right) \left\langle \varphi_j' \middle| \varphi_j \right\rangle \left\langle \psi_j \middle| \varphi_j' \right\rangle \left\langle \varphi_j \middle| \psi_j \right\rangle d\varphi_j' d\varphi_j$$

$$= \int \int \left(\varphi_j - \left\langle \hat{\varphi}_j \right\rangle\right)^2 \delta\left(\varphi_j' - \varphi_j\right) \left\langle \varphi_j' \middle| \psi_j \right\rangle^* \left\langle \varphi_j \middle| \psi_j \right\rangle d\varphi_j' d\varphi_j$$

$$= \int \left(\varphi_j - \nu\right)^2 \left|\left\langle \varphi_j \middle| \psi_j \right\rangle\right|^2 d\varphi_j.$$

Therefore, $\left\langle \psi_j \middle| \left(\hat{\Delta}\varphi_j\right)^2 \middle| \psi_j \right\rangle$ is the variance and $\sqrt{\left\langle \psi_j \middle| \left(\hat{\Delta}\varphi_j\right)^2 \middle| \psi_j \right\rangle}$ is the standard deviation.
Q.E.D.

The properties of a logical variable in a probabilistic inference system can be more completely specified by combining the probabilistic information resident in its truth state vector with the associated position probability distribution information contained in a position amplitude vector. This combination can be achieved for each variable in one of two ways to create either a π state or an ε state. We again emphasize that such states are of particular value for physical implementations of Dirac networks.

3.2 π STATES

The π state for the jth logical variable is the tensor product state $\left|\theta_j\right\rangle \left|\psi_j\right\rangle$. The π state $\left|\Phi\right\rangle$ for an n-variable inference system is the tensor product

$$\left|\Phi\right\rangle = \left|\theta_1\right\rangle \left|\psi_1\right\rangle \left|\theta_2\right\rangle \left|\psi_2\right\rangle \cdots \left|\theta_n\right\rangle \left|\psi_n\right\rangle.$$

Lemma 43 *If each position amplitude in a π state is normalized, then the π state is normalized.*

Proof: Since each position amplitude and truth state is normalized, then

$$\left\langle \Phi \middle| \Phi \right\rangle = \left\langle \theta_1 \middle| \theta_1 \right\rangle \left\langle \psi_1 \middle| \psi_1 \right\rangle \cdots \left\langle \theta_n \middle| \theta_n \right\rangle \left\langle \psi_n \middle| \psi_n \right\rangle = 1 \cdot 1 \cdots 1 \cdot 1 = 1.$$

Q.E.D.

3.3 THE MOMENTUM OPERATOR FOR A LOGICAL VARIABLE AND THE RIGID TRANSLATION OF TRUTH DISTRIBUTION FUNCTIONS

Complications arise when a variable is influenced by other variables or has evolved in time. In this case, the rotation and/or time evolution of the variable's prior truth state induced by the action of the associated rotation and/or time evolution operators must be accompanied by a consistent change in the position amplitude vector. The purpose of this section is to introduce a translation operator whose action will produce such a change in a position amplitude vector.

To formally define this operator, it is first necessary to assume that a ("fictitious" and necessarily) Hermitean *momentum operator* $\hat{p}_j$ can be assigned to the jth variable in an inference system and that it obeys the commutation axioms given by:

$$[\hat{p}_j, \hat{p}_k] = 0$$

and

$$\left[\hat{\varphi}_j, \hat{p}_k\right] = i\delta_{jk}.$$

Whenever the last axiom applies, it is a simple task to evaluate commutators of the form $\left[\hat{\varphi}_j, F\left(\hat{p}_j\right)\right]$, where $F\left(\hat{p}_j\right)$ is any function of the operator $\hat{p}_j$. Such an evaluation will be needed for determining the properties of the translation operator and can be readily obtained using the following theorem [2](since this theorem is well-established, it is stated here without proof):

Theorem 44 $\left[\hat{\varphi}_j, F\left(\hat{p}_j\right)\right] = i\frac{\partial F}{\partial \hat{p}_j}.$

The change of the position amplitude vector $|\psi_k\rangle$ in position space for the kth logical variable that is consistent with the *total cumulative rotation and time evolution* ξ_k of its truth state vector $|\theta_k\rangle$ induced by the action of some $\hat{\partial}_\sigma[\mathcal{G}]$ is achieved by the action upon $|\psi_k\rangle$ of the exponential *translation operator* defined by

$$\hat{S}_k\left(\beta_k\right) \equiv e^{-i\beta_k \hat{p}_k},$$

where β_k is the distance in position space that corresponds to the cumulative angular quantity ξ_k. The next theorem defines the action of this operator upon a position amplitude vector in position space.

Theorem 45 $\langle\varphi_k|\,\hat{S}_k\left(\beta_k\right)|\psi_k\rangle = \langle\varphi_k - \beta_k|\,\psi_k\rangle\,.$

Proof: Since the translation operator is an exponential function of $\hat{p}_k$, the previous theorem may be applied to yield

$$
\begin{aligned}
\left[\hat{\varphi}_k, \hat{S}_k\left(\beta_k\right)\right] &= \hat{\varphi}_k\hat{S}_k\left(\beta_k\right) - \hat{S}_k\left(\beta_k\right)\hat{\varphi}_k = i\frac{\partial\hat{S}_k\left(\beta_k\right)}{\partial\hat{p}_k} \\
&= i(-i\beta_k)\hat{S}_k\left(\beta_k\right) \\
&= \beta_k\hat{S}_k\left(\beta_k\right)
\end{aligned}
$$

so that

$$\hat{\varphi}_k\hat{S}_k\left(\beta_k\right) = \hat{S}_k\left(\beta_k\right)\left\{\hat{\varphi}_k + \beta_k\right\}.$$

Then

$$
\begin{aligned}
\hat{\varphi}_k\hat{S}_k\left(\beta_k\right)|\varphi_k\rangle &= \hat{S}_k\left(\beta_k\right)\left\{\hat{\varphi}_k + \beta_k\right\}|\varphi_k\rangle \\
&= \hat{S}_k\left(\beta_k\right)\left\{\varphi_k + \beta_k\right\}|\varphi_k\rangle \\
&= \left\{\varphi_k + \beta_k\right\}\hat{S}_k\left(\beta_k\right)|\varphi_k\rangle\,.
\end{aligned}
$$

This implies that

$$\hat{S}_k\left(\beta_k\right)|\varphi_k\rangle = |\varphi_k + \beta_k\rangle$$

or equivalently

$$\langle\varphi_k|\,\hat{S}_k^\dagger\left(\beta_k\right) = \langle\varphi_k + \beta_k|\,.$$

Since $\hat{S}_k^\dagger (\beta_k) = e^{i\beta_k \hat{p}_k^\dagger} = e^{i\beta_k \hat{p}_k}$, then $\hat{S}_k^\dagger (-\beta_k) = \hat{S}_k (\beta_k)$. Using this in the last equation gives

$$\langle \varphi_k | \hat{S}_k^\dagger (-\beta_k) = \langle \varphi_k | \hat{S}_k (\beta_k) = \langle \varphi_k - \beta_k | \, .$$

Forming the scalar product with $|\psi_k\rangle$ gives the desired result:

$$\langle \varphi_k | \hat{S}_k (\beta_k) |\psi_k\rangle = \langle \varphi_k - \beta_k | \, \psi_k \rangle \, .$$

Q.E.D.

Thus, the projection of $\hat{S}_k (\beta_k) |\psi_k\rangle$ into its position space produces a wave function $\langle \varphi_k - \beta_k | \, \psi_k \rangle$ which is the rigid translation of $\langle \varphi_k | \, \psi_k \rangle$ (along the φ_k axis in the graph of $\langle \varphi_k | \, \psi_k \rangle$) by β_k. In our development, the specification of a π state *requires* that for *every* variable k which experiences a rotation and/or a time evolution there is a *single* translation operator $\hat{S}_k(\beta_k)$ that is applied to its probability amplitude vector $|\psi_k\rangle$ which rigidly translates it along the φ_k axis a distance β_k that corresponds to the *total* angular rotation and/or time evolution ξ_k experienced by the associated prior truth state $|\theta_k\rangle$). Such a specification is said to be *consistent* and the associated string of rotation, time evolution, and translation operators is a *consistent string*. Clearly, the magnitude of β_k remains constrained by the door-stop rule requirement for ξ_k.

Consider now the unitarity of the translation operator and the effect of its action upon the normalization of $|\psi_k\rangle$.

Lemma 46 $\hat{S}_k (\beta_k)$ *is unitary.*

Proof: Because $\hat{p}_k$ is Hermitean, we can write

$$\hat{S}_k^\dagger (\beta_k) \, \hat{S}_k (\beta_k) = e^{i\beta_k \hat{p}_k^\dagger} e^{-i\beta_k \hat{p}_k} = e^{i\beta_k \hat{p}_k} e^{-i\beta_k \hat{p}_k} \, .$$

Since $[\hat{p}_k, \hat{p}_k] = 0$, then Glauber's theorem applies so that

$$\hat{S}_k^\dagger (\beta_k) \, \hat{S}_k (\beta_k) = e^{i\beta_k (\hat{p}_k - \hat{p}_k)} = e^{\hat{0}_k} = \hat{1}_k \, .$$

Similarly for $\hat{S}_k (\beta_k) \, \hat{S}_k^\dagger (\beta_k)$. **Q.E.D.**

Lemma 47 *If $|\psi_k\rangle$ is normalized, then so is $\hat{S}_k (\beta_k) |\psi_k\rangle$.*

Proof: This is a consequence of the unitarity of the translation operator:

$$1 = \langle \psi_k | \, \psi_k \rangle = \langle \psi_k | \hat{1}_k |\psi_k\rangle = \langle \psi_k | \hat{S}_k^\dagger (\beta_k) \, \hat{S}_k (\beta_k) |\psi_k\rangle \, ,$$

or equivalently

$$
\begin{aligned}
1 &= \langle \psi_k | \hat{S}_k^\dagger (\beta_k) \, \hat{1}_k \hat{S}_k (\beta_k) |\psi_k\rangle \\
&= \int \langle \psi_k | \hat{S}_k^\dagger (\beta_k) |\varphi_k\rangle \langle \varphi_k | \hat{S}_k (\beta_k) |\psi_k\rangle \, d\varphi_k \\
&= \int \langle \varphi_k | \hat{S}_k (\beta_k) |\psi_k\rangle^* \langle \varphi_k | \hat{S}_k (\beta_k) |\psi_k\rangle \, d\varphi_k
\end{aligned}
$$

$$= \int \left| \langle \varphi_k | \, \hat{S}_k \left(\beta_k \right) | \psi_k \rangle \right|^2 d\varphi_k$$

$$= \int \left| \langle \varphi_k - \beta_k | \, \psi_k \rangle \right|^2 d\varphi_k.$$

Q.E.D.

Thus, the normalization of a position amplitude vector is preserved under the action of the translation operator. The following theorem describes the properties of truth distribution functions when such translations are involved.

Theorem 48 *If* $\left| \langle \varphi_k | \, \psi_k \rangle \right|^2$ *is a normalized truth distribution function with mean* ν *and standard deviation* s_k, *then* $\left| \langle \varphi_k | \, \hat{S}_k \left(\beta_k \right) | \psi_k \rangle \right|^2$ *is a rigid translation of* $\left| \langle \varphi_k | \, \psi_k \rangle \right|^2$ *with mean* $\nu - \beta_k$ *and standard deviation* s_k.

Proof: Since translation operators perform rigid translations, then $\left| \langle \varphi_k | \, \hat{S}_k \left(\beta_k \right) | \psi_k \rangle \right|^2$ is a rigid translation (along the φ_k axis) of $\left| \langle \varphi_k | \, \psi_k \rangle \right|^2$ and, therefore, has the same standard deviation as $\left| \langle \varphi_k | \, \psi_k \rangle \right|^2$. Since the translation is along the φ_k axis, its mean is shifted from that of the untranslated truth distribution function by the amount β_k. **Q.E.D.**

Before closing this section, it is important to note that if

$$|\mu_k\rangle \equiv \hat{S}_k \left(\beta_k \right) | \psi_k \rangle,$$

then

$$\langle \mu_k | \, \hat{\varphi}_k \, | \mu_k \rangle = \langle \psi_k | \, \hat{S}_k^\dagger \left(\beta_k \right) \hat{\varphi} \hat{S}_k \left(\beta_k \right) | \psi_k \rangle$$

is the expected position of the translated position amplitude $|\mu_k\rangle$. Therefore, in general, $\langle \hat{\varphi}_k \rangle$ can represent the expected position for either an independent (untranslated) variable or a translated variable. It will be clear from the context in which this notation is used whether one or the other or both possibilities apply.

3.4 THE π STATE FOR A GENERAL INFERENCE SYSTEM

Even though the order of application of rotation and time evolution operators is important for the specification of truth states, this is not the case for translation operators. This follows from the fact that we employ only one translation operator for each variable that is acted upon by rotation and/or time evolution operators and each such translation operator is applied to the associated position amplitude vector only after all rotation and time evolution operators have been appropriately applied to their associated truth state. Consequently, the following commutation relations hold for translation operators:

Lemma 49

$$\left[\hat{S}_j \left(\beta_j \right), \hat{S}_k \left(\beta_k \right) \right] = 0,$$

and similarly for the associated adjoint operators.

Proof: This follows directly from the fact that the momentum operator is Hermitean and that $[\hat{p}_j, \hat{p}_l] = 0$. **Q.E.D.**

Also note that since the influence and momentum operators are defined for distinct vector spaces, then $[\hat{\sigma}_j, \hat{p}_k] = 0$ so that all commutators between rotation and translation operators and their adjoints, as well as between time evolution and translation operators and their adjoints, vanish. Thus, consistent strings can always be expressed as a catenation $\mathcal{G}\mathcal{S}$ of a string of its rotation and/or time evolution operators $\mathcal{G}$ with that of its translation operators $\mathcal{S}$. The ordering operator $\hat{\partial}_\sigma [\,]$ can be applied to such a consistent string with the following useful result (the order of application of translation operators is unimportant since they are applied after $\hat{\partial}_\sigma [\mathcal{G}]$):

$$\hat{\partial}_\sigma [\mathcal{G}\mathcal{S}] = \hat{\partial}_\sigma [\mathcal{G}]\,\mathcal{S}.$$

The π state vector $|\Phi\rangle$ for an inference system with both rotational influences, time evolutions, and associated rigid translations can be written in the general form

$$|\Phi\rangle = \hat{\partial}[\mathcal{G}\mathcal{S}]\,|\Theta\rangle\,|\Psi\rangle = \hat{\partial}[\mathcal{G}]\mathcal{S}\,|\Theta\rangle\,|\Psi\rangle = \hat{\partial}[\mathcal{G}]\,|\Theta\rangle\,\mathcal{S}\,|\Psi\rangle,$$

where $|\Theta\rangle$ and $|\Psi\rangle$ are tensor products of all independent and prior truth states and position amplitudes, respectively. We shall refer to such a state as a *general π state* and the system it represents as a *general π inference system*. Observe that the last expression may also be written as

$$|\Phi\rangle = |\Xi\rangle\,\hat{\partial}[\mathcal{G}]\,|\Upsilon\rangle\,|\Lambda\rangle\,\mathcal{S}\,|\Omega\rangle,$$

where $|\Xi\rangle$ and $|\Lambda\rangle$ are the tensor products of the truth states and of the position amplitudes for all the independent variables, respectively, and $|\Upsilon\rangle$ and $|\Omega\rangle$ are the tensor products of the prior truth states and position amplitudes for all the variables effected by rotation and/or time evolution operators. A normalization guarantee for general π states is provided by the next theorem.

Theorem 50 *If each position amplitude vector in a general π state is normalized, then that general π state is normalized.*

Proof: Let $\mathcal{S} = \hat{S}_k\,(\beta_k) \cdots \hat{S}_m\,(\beta_m)$. Since truth states are always normalized and from the premise

$$\langle \Lambda |\,\Lambda \rangle = \langle \Omega |\,\Omega \rangle = 1,$$

then the unitary properties of $\hat{\partial}[\mathcal{G}]$ and the translation operators yields

$$
\begin{aligned}
\langle \Phi |\,\Phi \rangle &= \langle \Xi |\,\langle \Upsilon |\,\left(\hat{\partial}[\mathcal{G}]\right)^\dagger \hat{\partial}[\mathcal{G}]\,|\Upsilon\rangle\,|\Xi\rangle \cdot \\
&\quad \langle \Lambda |\,\langle \Omega |\,\hat{S}_m^\dagger\,(\beta_m) \cdots S_k^\dagger\,(\beta_k)\,\hat{S}_k\,(\beta_k) \cdots \hat{S}_m\,(\beta_m)\,|\Omega\rangle\,|\Lambda\rangle \\
&= \langle \Xi |\,\langle \Upsilon |\,\hat{1}\,|\Upsilon\rangle\,|\Xi\rangle\,\langle \Lambda |\,\langle \Omega |\,\hat{1}_m \cdots \hat{1}_k\,|\Omega\rangle\,|\Lambda\rangle \\
&= \langle \Xi |\,\langle \Upsilon |\,\Upsilon \rangle\,|\Xi\rangle\,\langle \Lambda |\,\langle \Omega |\,\Omega \rangle\,|\Lambda\rangle \\
&= \langle \Xi |\,\Xi \rangle\,\langle \Lambda |\,\Lambda \rangle \\
&= 1.
\end{aligned}
$$

Q.E.D.

3.5 π STATE AUTOCORRELATION FUNCTIONS

Consider the general π states given by

$$|\Phi_\sigma\rangle = |\Xi\rangle \, \hat{\partial}_\sigma[\mathcal{G}] \, |\Upsilon\rangle \, |\Lambda\rangle \, \mathcal{S} \, |\Omega\rangle$$

and

$$|\Phi'_v\rangle = |\Xi\rangle \, \hat{\partial}_v[\mathcal{G}'] \, |\Upsilon\rangle \, |\Lambda\rangle \, \mathcal{S}' \, |\Omega\rangle \, ,$$

where $|\Xi\rangle$, $|\Upsilon\rangle$, $|\Lambda\rangle$, and $|\Omega\rangle$ are as defined in the last section. The associated π state autocorrelation function is

$$
\begin{aligned}
\langle \Phi'_v | \, \Phi_\sigma \rangle &= \langle \Upsilon | \left(\hat{\partial}_v[\mathcal{G}'] \right)^\dagger \hat{\partial}_\sigma[\mathcal{G}] \, |\Upsilon\rangle \cdot \langle \Omega | \, (\mathcal{S}')^\dagger \, \mathcal{S} \, |\Omega\rangle \\
&= \mathcal{E} \cdot \mathcal{D}.
\end{aligned}
$$

Thus, a π state autocorrelation function factors into the product of a truth state autocorrelation function $\mathcal{E}$ and a position amplitude autocorrelation function $\mathcal{D}$.

Let us now evaluate each of these factors. If

$$\hat{\partial}_\sigma[\mathcal{G}] \, |\Upsilon\rangle = \hat{\partial}_\sigma \, [\mathcal{G}] \sum_{\omega \in L_N} f(\omega) \, |\omega\rangle$$

and

$$\hat{\partial}_v[\mathcal{G}'] \, |\Upsilon\rangle = \hat{\partial}_v \, [\mathcal{G}'] \sum_{\omega \in L_N} f(\omega) \, |\omega\rangle \, ,$$

then

$$
\begin{aligned}
\mathcal{E} &= \left[\sum_{\omega' \in L_N} f_v(\omega') \, \langle \omega' | \right] \left[\sum_{\omega \in L_N} f_\sigma(\omega) \, |\omega\rangle \right] \\
&= \sum_{\omega \in L_N} f_v(\omega) f_\sigma(\omega),
\end{aligned}
$$

where f_σ and f_v are f evaluated for the post-state angles obtained after the applications of $\hat{\partial}_\sigma[\mathcal{G}]$ and $\hat{\partial}_v[\mathcal{G}']$, respectively.

To determine $\mathcal{D}$ assume that

$$\mathcal{S} \, |\Omega\rangle = \hat{S}_1(\beta_1) \, |\psi_1\rangle \, \hat{S}_2(\beta_2) \, |\psi_2\rangle \cdots \hat{S}_n(\beta_n) \, |\psi_n\rangle$$

and

$$\mathcal{S}' \, |\Omega\rangle = \hat{S}'_1(\beta'_1) \, |\psi_1\rangle \, \hat{S}'_2(\beta'_2) \, |\psi_2\rangle \cdots \hat{S}'_n(\beta'_n) \, |\psi_n\rangle \, .$$

Then

$$\mathcal{D} = \mathcal{D}_1 \cdot \mathcal{D}_2 \cdots \mathcal{D}_n,$$

where

$$\begin{aligned}
\mathcal{D}_j &= \left\langle \psi_j \middle| \hat{S}_j'^\dagger(\beta_j')\hat{S}_j(\beta_j) \middle| \psi_j \right\rangle \\
&= \int \left\langle \psi_j \middle| \hat{S}_j'^\dagger(\beta_j') \middle| \varphi_j \right\rangle \left\langle \varphi_j \middle| \hat{S}_j(\beta_j) \middle| \psi_j \right\rangle d\varphi_j \\
&= \int \left\langle \varphi_j - \beta_j' \middle| \psi_j \right\rangle^* \left\langle \varphi_j - \beta_j \middle| \psi_j \right\rangle d\varphi_j, 1 \le j \le n
\end{aligned}$$

is the position amplitude autocorrelation function for the jth variable. Thus, $\mathcal{D}$ is the product of position amplitude autocorrelation functions for all the logical variables defined by $|\Omega\rangle$ (or $|\Upsilon\rangle$ since in either case the logical variables are the same).

3.6 POSITION MEASUREMENTS FOR GENERAL π INFERENCE SYSTEMS

As with truth operators, the position operators for the position amplitudes associated with the logical variables of a general π inference system can be used to define generalized position and uncertainty operators for any subset of system variables. If N is the set of variable indices for an inference system, then let $\emptyset \ne B \subseteq N$ define a subsystem with truth state vector space dimension $2^{|B|}$. The *generalized position operator* $\hat{P}_B$ for the inference subsystem with variables B is

$$\hat{P}_B \equiv \frac{1}{|B|} \sum_{j \in B} \hat{\varphi}_j. \tag{3-1}$$

The *generalized positional uncertainty operator* for the inference subsystem defined by B is

$$\hat{U}_B \equiv \left(\frac{1}{|B|}\right) \sum_{j \in B} \left(\hat{\Delta}\varphi_j\right)^2. \tag{3-2}$$

The expected position and an associated expected value for the variance for any subsystem of a general π inference system can be evaluated using the next theorem.

Theorem 51 *If $|\Phi\rangle$ is the general π state for a general π inference system with a subsystem defined by the variable index set B, then*

$$\langle\Phi| \hat{P}_B |\Phi\rangle = \frac{1}{|B|} \sum_{j \in B} \left\langle \hat{\varphi}_j \right\rangle,$$

and

$$\langle\Phi| \hat{U}_B |\Phi\rangle = \left(\frac{1}{|B|}\right) \sum_{j \in B} \left\langle \psi_j \middle| \left(\hat{\Delta}\varphi_j\right)^2 \middle| \psi_j \right\rangle.$$

Proof: Let N index the system variables and let $|\Phi\rangle = |\Theta\rangle |\beta\rangle |\lambda\rangle$, where $|\Theta\rangle$ is the tensor product of all truth state vectors,

$$|\beta\rangle = \oplus_{l \in N-B} |\chi_l\rangle$$

and

$$|\lambda\rangle = \oplus_{k \in B} |\mu_k\rangle.$$

Here $\oplus$ denotes tensor products and $|\chi_l\rangle$ and $|\mu_k\rangle$ represent position amplitudes for both independent variables (e.g., $|\chi_r\rangle = |\psi_r\rangle$) and translated amplitudes (e.g., $|\chi_r\rangle = \hat{S}_r(\beta_r)|\psi_r\rangle$). Then

$$
\begin{aligned}
\langle\Phi|\,\hat{P}_B\,|\Phi\rangle &= \langle\Theta|\,\langle\beta|\,\langle\lambda|\,\hat{P}_B\,|\lambda\rangle\,|\beta\rangle\,|\Theta\rangle \\
&= \langle\Theta|\,\Theta\rangle\,\langle\beta|\,\beta\rangle\,\langle\lambda|\,\hat{P}_B\,|\lambda\rangle \\
&= \langle\lambda|\,\hat{P}_B\,|\lambda\rangle \\
&= (\oplus_{k \in B}\langle\mu_k|)\left(\frac{1}{|B|}\sum_{j \in B}\hat{\varphi}_j\right)(\oplus_{k \in B}|\mu_k\rangle) \\
&= \frac{1}{|B|}\sum_{j \in B}\left\{\langle\mu_j|\,\hat{\varphi}_j\,|\mu_j\rangle\left(\Pi_{k \in B-\{j\}}\langle\mu_k|\,\mu_k\rangle\right)\right\} \\
&= \frac{1}{|B|}\sum_{j \in B}\langle\mu_j|\,\hat{\varphi}_j\,|\mu_j\rangle \\
&= \frac{1}{|B|}\sum_{j \in B}\langle\hat{\varphi}_j\rangle.
\end{aligned}
$$

The result for $\langle\Phi|\,\hat{U}_B\,|\Phi\rangle$ follows from Theorem 48. **Q.E.D.**

Therefore, when the position amplitudes for a general π inference system are normalized, it is a straightforward matter to evaluate the average mean position and associated uncertainty for a subsystem in terms of the known (translated and untranslated) means and uncertainties associated with each subsystem variable. Also, it is important to notice from this theorem that ; (i) the values for $\langle\Phi|\,\hat{P}_B\,|\Phi\rangle$ and $\langle\Phi|\,\hat{U}_B\,|\Phi\rangle$ do not depend upon the truth states; (ii) if $\left|\langle\varphi_j|\,\psi_j\rangle\right|^2$ represents the probability distribution for the truth value of variable j with $\langle\varphi_j\rangle = \langle\hat{\tau}_j\rangle$, then $\langle\Phi|\,\hat{P}_B\,|\Phi\rangle = \langle\hat{T}_B\rangle$; and (iii) $\langle\Phi|\,\hat{U}_B\,|\Phi\rangle$ provides a measure of the truth uncertainty. Consequently, it is possible to formally represent a Dirac network strictly in terms of position amplitude vectors.

3.7 ε STATES

3.7.1 The ε StateFor A Logical Variable

An ε state for the jth logical variable is the entangled state

$$\left|\theta_j; \psi_j^{\pm}\right\rangle = \cos\theta_j\,|+\rangle_j\left|\psi_j^+\right\rangle + \sin\theta_j\,|-\rangle_j\left|\psi_j^-\right\rangle, \tag{3-3}$$

where the $\left|\psi_j^{\pm}\right\rangle$ are normalized - but not necessarily orthogonal - position probability amplitudes such that $\left|\langle\varphi_j|\,\psi_j^{\pm}\rangle\right|^2 d\varphi_j$ are the probabilities for finding the distinguished positions $\varphi_j^{\pm}$ in the

interval $d\varphi_j$, respectively. These distinguished positions are associated with properties $v^{\pm}$ of the ± 1 truth values for the jth variable and the functions $\left|\left\langle\varphi_j\middle|\psi_j^{\pm}\right\rangle\right|^2$ are probability distributions for these properties with variances

$$\left\langle\psi_j^{\pm}\middle|\left(\hat{\Delta}\varphi_j\right)^2\middle|\psi_j^{\pm}\right\rangle = \lambda_j^{\pm} \tag{3-4}$$

and mean values

$$\left\langle\psi_j^{\pm}\middle|\hat{\varphi}_j\middle|\psi_j^{\pm}\right\rangle = \varphi_j^{\pm}. \tag{3-5}$$

Thus, the probability distribution function for the ε state of Equation(3-3) is the weighted sum of these distributions given by

$$\left|\left\langle\varphi_j\middle|\theta_j;\psi_j^{\pm}\right\rangle\right|^2 = \cos^2\theta_j\left|\left\langle\varphi_j\middle|\psi_j^{+}\right\rangle\right|^2 + \sin^2\theta_j\left|\left\langle\varphi_j\middle|\psi_j^{-}\right\rangle\right|^2 \tag{3-6}$$

and may be interpreted as the probability distribution for finding the associated system in a state described by the distinguished position φ_j^+ *or* the distinguished position φ_j^- when the events associated with the distributions are independent.

3.7.2 The ε State For A General Inference System

Using previously developed notation (see Equations (2-1), (2-2), and (2-3)), the *general ε state* $|\Psi\rangle$ for a *general* (n-variable) *ε inference system* is the entangled state

$$
\begin{aligned}
|\Psi_\sigma\rangle &= \hat{\partial}_\sigma[\mathcal{G}]\left|\theta_1;\psi_1^{\pm}\right\rangle\left|\theta_2;\psi_2^{\pm}\right\rangle\cdots\left|\theta_n;\psi_n^{\pm}\right\rangle \\
&= \hat{\partial}_\sigma[\mathcal{G}]\sum_{\omega\in L_N}f(\omega)|\omega\rangle|\psi_\omega\rangle \\
&= \sum_{\omega\in L_N}f_\sigma(\omega)|\omega\rangle|\psi_\omega\rangle,
\end{aligned}
\tag{3-7}
$$

where $f_\sigma(\omega)$ is $f(\omega)$ evaluated according to Equation(2-3) using the post-state angles induced by $\hat{\partial}_\sigma[\mathcal{G}]$ and

$$|\psi_\omega\rangle \equiv |\psi_{\omega_1\omega_2\cdots\omega_n}\rangle = |\psi_1^{\omega_1}\rangle|\psi_2^{\omega_2}\rangle\cdots|\psi_n^{\omega_n}\rangle$$

is the amplitude associated with the probability of finding the distinguished positions $\varphi_1^{\omega_1}$ in $d\varphi_1, \varphi_2^{\omega_2}$ in $d\varphi_2, \cdots$, and $\varphi_n^{\omega_n}$ in $d\varphi_n$. Observe that only strings $\mathcal{G}$ of rotation and/or time evolution operators appear in this definition. Translation operators are not used in order that the amplitudes $\left|\psi_j^{\pm}\right\rangle$ remain fixed in their position spaces. The probability distribution function for Equation(3-7) is the *weighted sum*

$$|\langle\varphi_\omega|\Psi\rangle|^2 = \sum_{\omega\in L_N}[f_\sigma(\omega)]^2|\langle\varphi_\omega|\psi_\omega\rangle|^2,$$

where

$$|\langle\varphi_\omega|\ \psi_\omega\rangle|^2 \equiv |\langle\varphi_1|\ \psi_1^{\omega_1}\rangle|^2\,|\langle\varphi_2|\ \psi_2^{\omega_2}\rangle|^2\cdots|\langle\varphi_n|\ \psi_n^{\omega_n}\rangle|^2$$

is the probability distribution function associated with finding the distinguished positions as described above.

Theorem 52 *A general ε state is normalized.*

Proof:

$$
\begin{aligned}
\langle\Psi_\sigma|\ \Psi_\sigma\rangle &= \left[\sum_{\omega'\in L_N} f(\omega')\,\langle\omega'|\,\langle\psi_{\omega'}|\right](\hat{\partial}_\sigma[\mathcal{G}])^\dagger\hat{\partial}_\sigma[\mathcal{G}]\left[\sum_{\omega\in L_N} f(\omega)\,|\omega\rangle\,|\psi_\omega\rangle\right] \\
&= \sum_{\omega'\in L_N}\sum_{\omega\in L_N} f(\omega')f(\omega)\,\langle\omega'\ |\omega\rangle\,\langle\psi_{\omega'}\ |\psi_\omega\rangle \\
&= \sum_{\omega'\in L_N}\sum_{\omega\in L_N} f(\omega')f(\omega)\delta_{\omega'\omega}\,\langle\psi_{\omega'}\ |\psi_\omega\rangle \\
&= \sum_{\omega\in L_N} [f(\omega)]^2\,\langle\psi_\omega\ |\psi_\omega\rangle \\
&= \sum_{\omega\in L_N} [f(\omega)]^2 \\
&= 1.
\end{aligned}
$$

Q.E.D.

3.7.3 ε State Autocorrelation Functions

Consider the general ε states given by

$$|\Psi_\sigma\rangle = \hat{\partial}_\sigma[\mathcal{G}]\sum_{\omega\in L_N} f(\omega)\,|\omega\rangle\,|\psi_\omega\rangle$$

and

$$|\Psi'_v\rangle = \hat{\partial}_v[\mathcal{G}']\sum_{\omega\in L_N} g(\omega)\,|\omega\rangle\,|\xi_\omega\rangle\,.$$

The associated ε state autocorrelation function is

$$
\begin{aligned}
\langle\Psi'_v|\ \Psi_\sigma\rangle &= \left[\sum_{\omega'\in L_N} g(\omega')\,\langle\omega'|\,\langle\xi_{\omega'}|\right]\left(\hat{\partial}_v\,[\mathcal{G}']\right)^\dagger\hat{\partial}_\sigma\,[\mathcal{G}]\left[\sum_{\omega\in L_N} f(\omega)\,|\omega\rangle\,|\psi_\omega\rangle\right] \\
&= \sum_{\omega\in L_N} [g_v(\omega)f_\sigma(\omega)]\,\langle\xi_\omega|\ \psi_\omega\rangle \\
&= \sum_{\omega\in L_N} [g_v(\omega)f_\sigma(\omega)]\int \langle\varphi_\omega|\ \xi_\omega\rangle^*\,\langle\varphi_\omega|\ \psi_\omega\rangle\,d\varphi_\omega,
\end{aligned}
$$

where

$$\int \langle \varphi_\omega | \xi_\omega \rangle^* \langle \varphi_\omega | \psi_\omega \rangle \, d\varphi_\omega \;=\; \int \langle \varphi_1 | \xi_1 \rangle^* \langle \varphi_1 | \psi_1 \rangle \, d\varphi_1 \int \langle \varphi_2 | \xi_2 \rangle^* \langle \varphi_2 | \psi_2 \rangle \, d\varphi_2$$

$$\cdots \int \langle \varphi_n | \xi_n \rangle^* \langle \varphi_n | \psi_n \rangle \, d\varphi_n.$$

3.7.4   Post-selected ε States And Conditional Probability Distributions

A post-selected ε state is one which has been projected onto one of its (possible) truth state vectors. Specifically, the $\left| \theta'_j \right\rangle$ post-selected ε state for the state given by Equation (3-3) is defined as the scalar product

$$\begin{aligned} \left| \theta'_j : \theta_j; \psi_j^\pm \right\rangle &\equiv \left\langle \theta'_j \middle| \theta_j; \psi_j^\pm \right\rangle & (3\text{-}8) \\ &= \left[\cos \theta'_{j\;j} \langle + | + \sin \theta'_{j\;j} \langle - | \right] \left| \theta_j; \psi_j^\pm \right\rangle \\ &= \cos \theta'_j \cos \theta_j \left| \psi_j^+ \right\rangle + \sin \theta'_j \sin \theta_j \left| \psi_j^- \right\rangle \end{aligned}$$

which can be interpreted as the conditional probability amplitude for finding variable j in truth state $\left| \theta'_j \right\rangle$ if it is in state $\left| \theta_j; \psi_j^\pm \right\rangle$. Projecting this state into its position space yields

$$\left\langle \varphi_j \middle| \theta'_j : \theta_j; \psi_j^\pm \right\rangle = \cos \theta'_j \cos \theta_j \left\langle \varphi_j \middle| \psi_j^+ \right\rangle + \sin \theta'_j \sin \theta_j \left\langle \varphi_j \middle| \psi_j^- \right\rangle,$$

so that the associated *conditional probability distribution function* is given by

$$\begin{aligned} \left| \left\langle \varphi_j \middle| \theta'_j : \theta_j; \psi_j^\pm \right\rangle \right|^2 &= \cos^2 \theta'_j \cos^2 \theta_j \left| \left\langle \varphi_j \middle| \psi_j^+ \right\rangle \right|^2 + & (3\text{-}9) \\ &\quad \sin^2 \theta'_j \sin^2 \theta_j \left| \left\langle \varphi_j \middle| \psi_j^- \right\rangle \right|^2 + \\ &\quad 2 \cos \theta'_j \cos \theta_j \sin \theta'_j \sin \theta_j \, \mathrm{Re} \left\langle \varphi_j \middle| \psi_j^+ \right\rangle^* \left\langle \varphi_j \middle| \psi_j^- \right\rangle. \end{aligned}$$

This may be interpreted to be the probability distribution function for finding (or measuring) a state's position values to be φ_j^+ or φ_j^- conditioned upon the fact that the truth state is $\left| \theta'_j \right\rangle$ when it is in state $\left| \theta_j; \psi_j^\pm \right\rangle$. From this it is readily seen that in addition to being a weighted sum of the $\left| \left\langle \varphi_j \middle| \psi_j^\pm \right\rangle \right|^2$ distributions- unlike the probability distribution of Equation (3-6) - the conditional probability distribution for a variable's post selected ε state includes *an interference term* (the third term, provided $\theta'_j, \theta_j \neq 0, \frac{\pi}{2}$). This term can significantly modify the associated distribution from that of a simple weighted sum of the $\left| \left\langle \varphi_j \middle| \psi_j^\pm \right\rangle \right|^2$ distributions. Also, notice that this interference term can be written equivalently as $\left(\frac{1}{2} \right) \Delta \tau_j \cdot \Delta \tau'_j \, \mathrm{Re} \left\langle \varphi_j \middle| \psi_j^+ \right\rangle^* \left\langle \varphi_j \middle| \psi_j^- \right\rangle$. *This clearly relates this interference to the truth uncertainties $\Delta \tau_j$ and $\Delta \tau'_j$ that are associated with $\left| \theta_j \right\rangle$ and $\left| \theta'_j \right\rangle$, respectively.*

The generalized $|\phi\rangle = \sum_{\omega \in L_N} g(\omega) |\omega\rangle$ post-selected ε state for the state $|\Psi_\sigma\rangle$ given by Equation(3-7) is

$$
\begin{aligned}
|\phi : \Psi_\sigma\rangle &\equiv \langle\phi|\ \Psi_\sigma\rangle \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3\text{-}10)\\
&= \left[\sum_{\omega' \in L_N} g(\omega') \langle\omega'| \right] \hat{\partial}_\sigma [\mathcal{G}] \left[\sum_{\omega \in L_N} f(\omega) |\omega\rangle |\psi_\omega\rangle \right]\\
&= \sum_{\omega \in L_N} [g(\omega) f_\sigma(\omega)] |\psi_\omega\rangle ,
\end{aligned}
$$

where $f_\sigma(\omega)$ is as defined above. By extension, this is the general conditional probability amplitude for finding the inference system in state $|\phi\rangle$ when it is in state $|\Psi_\sigma\rangle$. The associated conditional probability distribution function is

$$
|\langle\varphi_\omega|\ \phi : \Psi_\sigma\rangle|^2 = \left| \sum_{\omega \in L_N} [g(\omega) f_\sigma(\omega)] \langle\varphi_\omega|\ \psi_\omega\rangle \right|^2 .
$$

Since this conditional probability distribution function is the square modulus of a sum, it exhibits *multiple interference terms*.

Thus - in a sense - post-selection *"erases"* specific truth state information from an ε state. The result of such a *loss of specific truth state information* is to produce an *interference "pattern"* in the associated distribution function which - as shown above - reflects the uncertainty in the truth values. This is in stark contrast to a non-post-selected (i.e. a non-erased) ε state which contains specific truth state information: its probability distribution is strictly a weighted sum which exhibits no such interference pattern.

3.7.5 Position Measurements For General ε Inference Systems

Let $\hat{P}_B$ and $\hat{U}_B$ be the previously defined (see Equations (3-1) - (3-2)) generalized position and generalized positional uncertainty operators, respectively. The expected values for these operators for a general ε inference system can be evaluated using the following theorems.

Theorem 53 *If $|\Psi_\sigma\rangle$ is the general ε state for a general ε inference system with a subsystem defined by the variable index set B, then*

$$
\langle\Psi_\sigma|\ \hat{P}_B |\Psi_\sigma\rangle = \frac{1}{|B|} \sum_{j \in B} \left[\varphi_j^+ \cos^2 \theta_j + \varphi_j^- \sin^2 \theta_j \right]
$$

and

$$
\langle\Psi_\sigma|\ \hat{U}_B |\Psi_\sigma\rangle = \frac{1}{|B|} \sum_{j \in B} \left[\lambda_j^+ \cos^2 \theta_j + \lambda_j^- \sin^2 \theta_j \right],
$$

where the θ_j's are post-state truth angles.

Proof: Let $N = \{1, 2, \cdots, n\}$ index the system variables and

$$
\begin{aligned}
|\Psi_\sigma\rangle &= \hat{\partial}_\sigma \left[\mathcal{G}\right] \Pi_{j \in N} \left|\phi_j; \psi_j^\pm\right\rangle \\
&= \Pi_{j \in N} \left|\theta_j; \psi_j^\pm\right\rangle \\
&= \left|\theta_1; \psi_1^\pm\right\rangle \left|\theta_2; \psi_2^\pm\right\rangle \cdots \left|\theta_n; \psi_n^\pm\right\rangle,
\end{aligned}
$$

where the θ_j's are the post-state angles induced by $\hat{\partial}_\sigma \left[\mathcal{G}\right]$. Then

$$
\begin{aligned}
\langle\Psi_\sigma| \hat{P}_B |\Psi_\sigma\rangle &= \frac{1}{|B|} \Pi_{k \in N} \left\langle\theta_k; \psi_k^\pm\right| \left[\sum_{j \in B} \hat{\varphi}_j\right] \Pi_{\ell \in N} \left|\theta_\ell; \psi_\ell^\pm\right\rangle \\
&= \frac{1}{|B|} \sum_{j \in B} \left\langle\theta_j; \psi_j^\pm\right| \hat{\varphi}_j \left|\theta_j; \psi_j^\pm\right\rangle \Pi_{k \in N-\{j\}} \left\langle\theta_k; \psi_k^\pm \middle| \theta_k; \psi_k^\pm\right\rangle \\
&= \frac{1}{|B|} \sum_{j \in B} \left\langle\theta_j; \psi_j^\pm\right| \hat{\varphi}_j \left|\theta_j; \psi_j^\pm\right\rangle \\
&= \frac{1}{|B|} \sum_{j \in B} \left[\cos\theta_{j \ j} \langle+| \left\langle\psi_j^+\right| + \sin\theta_{j \ j} \langle-| \left\langle\psi_j^-\right|\right] \hat{\varphi}_j \cdot \\
&\qquad \left[\cos\theta_j |+\rangle_j \left|\psi_j^+\right\rangle + \sin\theta_j |-\rangle_j \left|\psi_j^-\right\rangle\right] \\
&= \frac{1}{|B|} \sum_{j \in B} \left[\cos^2\theta_j \left\langle\psi_j^+\right| \hat{\varphi}_j \left|\psi_j^+\right\rangle + \sin^2\theta_j \left\langle\psi_j^-\right| \hat{\varphi}_j \left|\psi_j^-\right\rangle\right] \\
&= \frac{1}{|B|} \sum_{j \in B} \left[\varphi_j^+ \cos^2\theta_j + \varphi_j^- \sin^2\theta_j\right],
\end{aligned}
$$

and similarly for the expected variance. **Q.E.D.**

Theorem 54 *If $|\phi : \Psi_\sigma\rangle$ is a post-selected ε state for a general ε inference system with variable index set N and a subsystem defined by the variable index set $B \subseteq N$, then*

$$
\langle\phi : \Psi_\sigma| \hat{P}_B |\phi : \Psi_\sigma\rangle = \frac{1}{|B|} \sum_{j \in B} [\left\langle\phi_j : \theta_j; \psi_j^\pm\right| \hat{\varphi}_j \left|\phi_j : \theta_j; \psi_j^\pm\right\rangle \cdot
$$
$$
\Pi_{k \in N-\{j\}} \left\langle\phi_k : \theta_k; \psi_k^\pm \middle| \phi_k : \theta_k; \psi_k^\pm\right\rangle]
$$

and

$$
\langle\phi : \Psi_\sigma| \hat{U}_B |\phi : \Psi_\sigma\rangle = \frac{1}{|B|} \sum_{j \in B} [\left\langle\phi_j : \theta_j; \psi_j^\pm\right| \left(\hat{\Delta}\varphi_j\right)^2 \left|\phi_j : \theta_j; \psi_j^\pm\right\rangle \cdot
$$
$$
\Pi_{k \in N-\{j\}} \left\langle\phi_k : \theta_k; \psi_k^\pm \middle| \phi_k : \theta_k; \psi_k^\pm\right\rangle],
$$

where the θ_j's and θ_k's are post-state truth angles,

$$
\left\langle\phi_k : \theta_k; \psi_k^\pm \middle| \phi_k : \theta_k; \psi_k^\pm\right\rangle = \cos^2\phi_k \cos^2\theta_k + \sin^2\phi_k \sin^2\theta_k +
$$

$$2 \cos \phi_k \cos \theta_k \sin \phi_k \sin \theta_k \operatorname{Re} \left\langle \psi_k^+ \middle| \psi_k^- \right\rangle$$

$$= \cos^2 \phi_k \cos^2 \theta_k + \sin^2 \phi_k \sin^2 \theta_k +$$

$$\left(\frac{1}{2} \right) \Delta \tau_{\phi_k} \cdot \Delta \tau_{\theta_k} \operatorname{Re} \left\langle \psi_k^+ \middle| \psi_k^- \right\rangle,$$

and

$$\left\langle \phi_j : \theta_j; \psi_j^\pm \middle| \hat{X}_j \middle| \phi_j : \theta_j; \psi_j^\pm \right\rangle = x_j^+ \cos^2 \phi_j \cos^2 \theta_j + x_j^- \sin^2 \phi_j \sin^2 \theta_j +$$

$$2 \cos \phi_j \cos \theta_j \sin \phi_j \sin \theta_j \operatorname{Re} \left\langle \psi_j^+ \middle| \hat{X}_j \middle| \psi_j^- \right\rangle$$

$$= x_j^+ \cos^2 \phi_j \cos^2 \theta_j + x_j^- \sin^2 \phi_j \sin^2 \theta_j +$$

$$\left(\frac{1}{2} \right) \Delta \tau_{\phi_j} \cdot \Delta \tau_{\theta_j} \operatorname{Re} \left\langle \psi_j^+ \middle| \hat{X}_j \middle| \psi_j^- \right\rangle$$

when $\hat{X}_j = \hat{\varphi}_j$ and $x_j^\pm = \varphi_j^\pm$ or $\hat{X}_j = \left(\hat{\Delta \varphi}_j \right)^2$ and $x_j^\pm = \lambda_j^\pm$ and where $\Delta \tau_{\phi_j}$ is the truth uncertainty for $\left| \phi_j \right\rangle$.

Proof: Consider

$$\hat{P}_B \left| \phi : \Psi_\sigma \right\rangle = \frac{1}{|B|} \sum_{j \in B} \hat{\varphi}_j \left| \phi_j : \theta_j; \psi_j^\pm \right\rangle \Pi_{k \in N - \{j\}} \left| \phi_k : \theta_k; \psi_k^\pm \right\rangle,$$

where the θ_j's and θ_k's are post-state truth angles. Then

$$\left\langle \phi : \Psi_\sigma \middle| \hat{P}_B \middle| \phi : \Psi_\sigma \right\rangle = \frac{1}{|B|} \left[\Pi_{\ell \in N} \left\langle \phi_\ell : \theta_\ell; \psi_\ell^\pm \middle| \right] \cdot$$

$$\left[\sum_{j \in B} \hat{\varphi}_j \left| \phi_j : \theta_j; \psi_j^\pm \right\rangle \Pi_{k \in N - \{j\}} \left| \phi_k : \theta_k; \psi_k^\pm \right\rangle \right]$$

$$= \frac{1}{|B|} \sum_{j \in B} [\left\langle \phi_j : \theta_j; \psi_j^\pm \middle| \hat{\varphi}_j \middle| \phi_j : \theta_j; \psi_j^\pm \right\rangle \cdot$$

$$\Pi_{k \in N - \{j\}} \left\langle \phi_k : \theta_k; \psi_k^\pm \middle| \phi_k : \theta_k; \psi_k^\pm \right\rangle],$$

and similarly for $\hat{U}_B$. The desired expressions for $\left\langle \phi_j : \theta_j; \psi_j^\pm \middle| \hat{X}_j \middle| \phi_j : \theta_j; \psi_j^\pm \right\rangle$, $\hat{X}_j \in \left\{ \hat{\varphi}_j, \hat{\Delta \varphi}_j \right\}$, and $\left\langle \phi_k : \theta_k; \psi_k^\pm \middle| \phi_k : \theta_k; \psi_k^\pm \right\rangle$ are readily obtained using Equations(3-8), (3-5), and (3-4) and theorem 4. **Q.E.D.**

4 REPRESENTATIVE EXAMPLE APPLICATIONS

The purpose of this section is to provide the reader with examples of how this Dirac algebra based method can be applied to assist in the solution of Bayesian inference, systems analysis, and data analysis problems.

4.1 BAYESIAN INFERENCING

In this first section we contrast a Dirac algebra based approach to Bayesian inferencing with contemporary methods. The Dirac truth state theory developed above will be applied to several representative problems in order to illustrate how the methodology can be used in this problem domain. These problems are: (1) a comparison of results obtained using the Dirac method with those obtained from the *Situational Influence Assessment Model* (SIAM) system for two very simple influence networks; (2) applying the Dirac method to determine the probability of successful engagement for a cruise missile seeking a mobile target; and (3) using the Dirac method to solve a "textbook" genetic disease problem. The Dirac based results were obtained using a prototype software system DIRACNET (see Appendix A) which implements the Dirac truth state theory developed above (the Dirac theory for probability distributions developed above has not been implemented in this software). In what follows, Dirac algebra based inference systems will be referred to as *Dirac networks* or - for short -as *Dirac nets*.

4.1.1 An Overview Of Automated Inferencing And Bayesian Networks

Since the 1960's, a variety of approaches to automated inferencing have been developed (e.g., decision trees, decision networks, rule-based systems, causal networks, and belief/Bayesian networks). Extensive discussion of the problems involved and approaches to their solution are available elsewhere [7, 8, 10]. These approaches are typically distinguishable by the form of logic/inference they employ (two-valued or multi-valued logics; deductive, inductive, or abductive inference; and temporal reasoning). The various problem domains to which these approaches have been applied can generally be classified as: categorical or probabilistic problems; explicit decision mechanisms or implicit decision making problems; and symbolic logic or numerical representation problems. Although not addressed here, experience has shown that certain of these approaches are better than others for addressing specific problem domain applications.

Early rule-based systems generally employed straightforward deterministic logic as their methodological basis. As the level of sophistication and experience increased, non-deterministic systems were developed which relied upon probabilistic and/or weighting schemes for their operation. These weighting schemes are either extensions of probabilistic methods or behave as probability measures when certain restrictions are imposed upon them. Consequently, those systems which are developed as belief - or causal - networks tend to be predominantly structured around a probability calculus as the foundation of their operation. Decision problems also employ probabilistic techniques and are often modeled as networks in which the topology

attempts to represent those significant factors and influences that are associated with the decision problem of interest. Decision trees are an example of this. Such trees are designed so that the leaf nodes of the tree topology serve as the final decision variables for the associated decision problem.

As mentioned above, such systems are also classified by the form of inference they employ for their operation. Let H be an hypothesis and ε be the associated evidence. Using this notation, the three types of inference can be easily defined as follows:

- *Deductive Inference*:

$$\text{If } H, \text{ then } \varepsilon.$$
$$\frac{H}{\text{Therefore, } \varepsilon.}$$

- *Inductive Inference*:

$$\text{Possibly } H \text{ accounts for } \varepsilon.$$
$$\frac{\varepsilon \text{ and } H, \varepsilon \text{ and } H, \cdots, \varepsilon \text{ and } H.}{\text{Therefore, if } H, \text{ then } \varepsilon.}$$

- *Abductive Inference*:

$$\text{If } H, \text{ then } \varepsilon.$$
$$\frac{\varepsilon}{\text{Therefore, } H.}$$

Here the hypothesis H may be true or false (or may acquire a value from a set of possible values) and the evidence ε can totally, partially, or not support the values for H. The simplest inference systems use only deductive inference, whereas more complex systems (e.g., belief nets) may use both deductive and abductive reasoning as their methods of inference.

Both causal and belief nets ideally organize their logical structure in the form of directed acyclic graph (DAG) representations (however, this may not always be the case or may not be possible for certain applications). Each node in the digraph represents a proposition and the arcs connecting them represent the influences of one proposition upon another. The proposition may be true or false - or it may assume one of a set of possible values. The evidence supplied to the system can support or deny at some level the propositions associated with the system nodes, as well as their influences upon other node propositions.

Such systems are initialized by assigning to each node a measure - typically a probabilistic measure - which reflects a prior belief or probability associated with each node (proposition). A similar measure which reflects the strength of the influence one node exerts upon another is assigned to each arc in the network. Using this, conditional probabilities that child nodes are true/false given that their parents are true/false are then determined. After initialization, the inferencing process can be invoked by instantiating the system, i.e. by establishing the truth or falsehood of one or more system propositions (nodes).

Algorithms are usually employed to control and update the probabilities in the network using π and λ message passing. The π values are determined from node and arc probability measures and are passed as π messages from parent nodes to child nodes in order to modify their probability measures. The λ messages are used to pass λ values from child nodes to parent nodes. For either of these cases, the system is ideally designed so that the updates at each node result only from the information received from adjacent nodes. In a system for which a DAG defines its topology (e.g. a directed tree), the message passing complexity is usually quite manageable. However, if the network topology which represents the inference system is not represented by a DAG - i.e. when the underlying digraph contains cycles - then updating can become extremely complex and cumbersome since it may involve re-evaluations of all joint probabilities for the system.

4.1.2 Dirac Networks As Models For Inference Systems

The structure and operation of Dirac networks is similar to that of Bayesian networks. In particular: (1) both employ digraphs to represent their proposition/influence topologies; and (2) both pass update information between nodes along directed arcs of influence. However, as discussed above, Dirac networks use angular measures for proposition truth state vectors (Figure 4.1-1) instead of probability measures to represent the truthfulness of propositions (nodes). Furthermore, the influences of nodes upon other nodes are obtained by modifying these angular measures via rotations of the associated truth state vectors. These rotations are accomplished through the ordered actions of unitary rotation operators upon the truth state vectors for the associated child variables and eliminates the need for probabilistic propagation of updates within the network.

For a Dirac network an initial estimate of the truthfulness for each node (proposition) in the underlying network is made by assigning an angle $0^0 \leq \theta \leq 90^0$ to define each of the associated truth state vectors. The effects of influences between nodes are contained within the rotation angles α that are assigned to each arc in the network. Unlike Bayesian networks - which require a probabilistic form for propagating influences (and updates), Dirac networks only utilize simple (ordered) vector rotations to achieve the update. In addition, *this rotation permits a level of flexibility that is not available in conventional Bayesian networks* - the rotation angle α can assume any functional form without concern for violating system normalization requirements. Either "prior" or "posterior" values for θ may be used when computing θ-dependent α angles via the proper utilization of order indexing for the associated rotation operators (when posterior θ's are used, the influencing nodes are updated before the influenced nodes). The use of order indexing yields a procedure that is analogous to λ- and π-message passing in belief networks.

Some of the major differences between Dirac networks and Bayesian-type networks are summarized in the following table.

Dirac Networks
oangular values assigned to nodes define truth as a vector
oorder indexed rotation operators propagate influences
opropagation model is unconstrained
oone-/bi-directional/cyclic propagation permitted
(one-directional demonstrated)

Bayesian Networks
oprior/conditional probabilities assigned to nodes
oλ and π messages propagate influences
oprobability calculus is required for propagation
oone-directional and bi-directional propagation is possible
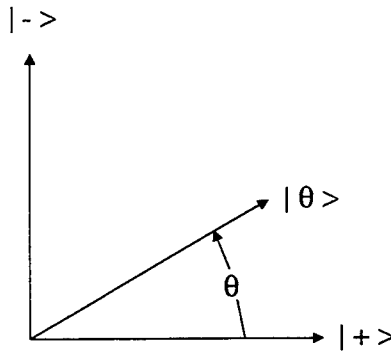
Figure 4.1-1. Truth Vector Space Showing Truth State For Variable θ

4.1.3 A Comparison With SIAM

As a first case, comparison is made between the results obtained using DIRACNET and the SIAM system [9] for the two very simple networks specified with their prior probability values in Figure 4.1-2 (net 1) and Figure 4.1-3 (net 2). SIAM uses prior probability assignments to network nodes and both a positive influence (labelled g in Figures 4.1-2 and 4.1-3) and a negative influence (labeled h in Figure 4.1-2 and 4.1-3) between nodes. These influence values are related to conditional probabilities and the positive (negative) influence expresses how likely it is that the influenced node will be changed if the influencing node is true (false). After the network has been processed, the nodes that are influenced will have updated marginal probability values.

The algebraic expression for the Dirac truth state for net 1 is (rotation operator application

order doesn't matter since the influence angles are constant):

$$|\Psi\rangle_{net\ 1} = \hat{R}_{AB}\left(29.7^0\right)\hat{R}_{BC}\left(11.7^0\right)\left|54_A^0\right\rangle\left|45_B^0\right\rangle\left|36_C^0\right\rangle$$
$$= \left|54_A^0\right\rangle\left|15.3_B^0\right\rangle\left|24.3_C^0\right\rangle.$$

Based upon this, DIRACNET yielded posterior probabilities of 0.4, 0.83, and 0.73 for nodes A, B, and C, respectively. These values agreed with the SIAM results to within the limits of numerical roundoff accuracy.

The algebraic expression for the Dirac truth state for net 2 is (again, application order doesn't matter):

$$|\Psi\rangle_{net\ 2} = \hat{R}_{AD}\left(9^0\right)\hat{R}_{BD}\left(9^0\right)\hat{R}_{CD}\left(9^0\right)\left|45_A^0\right\rangle\left|45_B^0\right\rangle\left|45_C^0\right\rangle\left|45_D^0\right\rangle$$
$$= \left|45_A^0\right\rangle\left|45_B^0\right\rangle\left|45_C^0\right\rangle\left|18_D^0\right\rangle.$$

DIRACNET provided posterior probabilities of 0.5, 0.5, 0.5, and 0.80 for nodes A, B, C, and D, respectively. The posterior probabilities generated by SIAM are 0.5, 0.5, 0.5, and 0.789 for nodes A, B, C, and D, respectively. This small discrepancy is likely due to integer roundoff used in DIRACNET calculations.
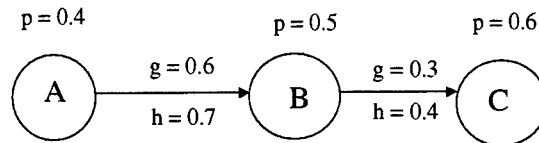
Figure 4.1-2. A Simple Probability Network
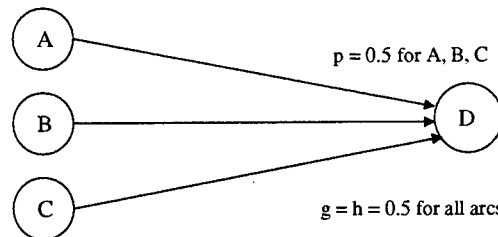
Figure 4.1-3. A Simple Probability Network

4.1.4   A Dirac Network For A Cruise Missile Search And Destroy Mission

This example shows how the Dirac method can be applied to problems of military interest. In order to illustrate this, a *simplified version* of a cruise missile search and destroy mission has

been selected. In particular, a cruise missile is assumed to be seeking a mobile target in varying terrain and variable weather conditions. The problem is to construct a Dirac network which will use observational evidence to predict the probability that the target will be successfully engaged by the missile. It is assumed that the search area is partitioned into search regions. The Dirac network employed in this illustration is designed to use only one of many possible search patterns available to a cruise missile in order to find a mobile target in one of these regions, i.e. region A. In practice, the search evaluation would clearly involve all regions in the partition and might involve determining the probabilities for engagement for a variety of possible search patterns that can be used by the missile in order to find the target. Slave Dirac networks for each region and search pattern combination would be concurrently processed in order to supply their associated engagement probabilities to a master Dirac network which would make the final recommendations for allocating missiles to search areas and the search patterns that would be best to use.

The logical variables and influences for the Dirac network in this example are shown in the schematic of Figure 4.1-4. There, integers juxtaposed to each logical assertion are the node labels for nodes associated with the assertions. Using this, we may readily write the following algebraic expression for this Dirac network's truth state vector:

$$|\Psi\rangle = \hat{\partial}[\mathcal{R}] \, |\theta_1\rangle \, |\theta_2\rangle \, |\theta_3\rangle \, |\theta_4\rangle \, |\theta_5\rangle \, |\theta_6\rangle \, |\theta_7\rangle \, |\theta_8\rangle \, |\theta_9\rangle \, |\theta_{10}\rangle \, |\theta_{11}\rangle \, ,$$

where the system influences are imposed by the composite rotation operator according to the ordering (order matters because θ_i-dependent rotation angles will be used)

$$
\begin{aligned}
\hat{\partial}[\mathcal{R}] \;=\; & \hat{R}_{10,11}\left(\alpha_{10,11}\right) \hat{R}_{9,10}\left(\alpha_{9,10}\right) \hat{R}_{8,9}\left(\alpha_{8,9}\right) \hat{R}_{7,10}\left(\alpha_{7,10}\right) \hat{R}_{6,8}\left(\alpha_{6,8}\right) \\
& \hat{R}_{5,7}\left(\alpha_{5,7}\right) \hat{R}_{4,9}\left(\alpha_{4,9}\right) \hat{R}_{4,7}\left(\alpha_{4,7}\right) \hat{R}_{3,6}\left(\alpha_{3,6}\right) \hat{R}_{2,8}\left(\alpha_{2,8}\right) \hat{R}_{2,5}\left(\alpha_{2,5}\right) \\
& \hat{R}_{1,5}\left(\alpha_{1,5}\right) .
\end{aligned}
$$

Here, θ_i is the prior angular setting for the ith variable (node) and $\alpha_{i,j}$ is the rotation angle for the influence of the ith variable upon the jth variable.

This is the *general truth state* manipulated and evaluated by DIRACNET. As a representative test case, the logical variables (nodes) for the network were initialized with angular settings according to the following table which also includes the prior expected truth values for

each variable:

i (node index)	θ_i (degrees)	$\langle \tau_i \rangle$ (prior value)
1	50	-0.174
2	40	0.174
3	0	1.000
4	30	0.500
5	45	0.000
6	25	0.643
7	45	0.000
8	45	0.000
9	45	0.000
10	45	0.000
11	45	0.000

Note that the node 3 angular setting of 0^0 insures that the assertion that "search pattern no. 1 is selected" is precisely true. Also, the assignment of $\theta_i = 45^0$ indicates that there is initially total uncertainty concerning the truth of the associated logical variables.

The influence angles $\alpha_{i,j}$ that were used are summarized in the next table. Included is a column which indicates how the influence angle was obtained from the value in the table. All but one of the influence angles ($\alpha_{3,6}$) that were applied via the associated rotation operators were obtained by using a scaled initial input value $\alpha_{i,j} = \gamma_i \cdot \alpha'_{i,j}$, where $\alpha'_{i,j}$ is the initial input value, $\alpha_{i,j}$ is the angle used in the rotation, and γ_i is a sigmoid scale factor given by

$$\gamma_i = \left\{ \frac{2}{1 + e^{-\beta(45^0 - \theta_i)}} - 0.5 \right\}.$$

Here, β is a factor which controls the slope of the transition region of the sigmoid function so that: (1) the closer θ_i is to 45^0 for the influencing node (i.e., the more uncertain it is), the closer the angle $\alpha_{i,j}$ is to 0^0; (2) if the influencing node is almost certainly false, the scale factor will be close to -1, thereby changing the sign of $\alpha_{i,j}$ with little impact upon its magnitude; and (3) if the influencing node is almost certainly true, the angle $\alpha_{i,j}$ will not change in sign or appreciably in magnitude. The use of such a scaled value is indicated by "scaled" in the "form of influence" column of the table. The remaining angle of influence is an unscaled application of the input value so that $\alpha_{3,6} = \alpha'_{3,6}$. This is denoted by "unscaled" in the "form of influence"

column.

i (parent index)	j (child index)	$\alpha'_{i,j}$ (degrees)	Form of Influence
1	5	10	scaled
2	5	10	scaled
2	8	10	scaled
3	6	10	unscaled
4	7	20	scaled
4	9	−10	scaled
5	7	20	scaled
6	8	10	scaled
7	10	10	scaled
8	9	10	scaled
9	10	10	scaled
10	11	10	scaled

Note the sign difference between $\alpha'_{4,7}$ and $\alpha'_{4,9}$. The purpose of this is to account for the fact that adequate cover increases the truth of the assertion that the "threat is in region A" but decreases the truth of the assertion that the "threat is detectable in region A."

Clearly, the order of rotation operator application is important for this problem since all but one influence angle is functionally dependent upon its associated parent truth state. It is obvious from a study of the direction of the arcs connecting the variables in Figure 4.1-4 and the order specification $\partial[\mathcal{R}]$ that the influences are propagating through the system starting from the variables at the bottom of the figure upwards to the top-most variable 11. The posterior values for each θ_i obtained after propagating the influences (via vector rotations) in one direction from the root nodes through the network to node 11 are given in the following table. Also tabulated are the posterior expected truth values for each logical variable.

i (node index)	θ_i (degrees)	$\langle \tau_i \rangle$ (posterior value)
1	50	−0.174
2	40	0.174
3	0	1.000
4	30	0.500
5	45	0.000
6	15	0.866
7	26	0.616
8	28	0.559
9	45	0.000
10	36	0.309
11	36	0.309

As can be calculated from this table, the "logic-based" certainty for the system provided by the system truth operator is $\langle \hat{T} \rangle = (\frac{1}{11}) \sum_{i=1}^{11} \langle \hat{\tau}_i \rangle = 0.378$. This suggests that additional

information about system variable truth states and influences which would increase the value for $\left\langle \hat{T} \right\rangle$ would be useful (this is always the case!). Also, as seen from this table, $\langle \tau_{10} \rangle = \langle \tau_{11} \rangle = 0.309$. This implies that the assertions "the threat is engaged" and "the mission is successful" are more true than false. If needed, the probabilities p_{10} and p_{11} for the truth of these assertions are easily computed from the posterior values for θ_{10} and θ_{11} using the theory developed above:

$$p_{10} = p_{11} = \cos^2 36^0 = 0.654.$$

Thus, based upon this formulation of the problem, if the probability of success p_{11} is acceptable to decision makers, then it is reasonable for them to allocate a missile to search region A using search pattern no. 1. Of course, evaluations based upon all other region/search pattern combinations might suggest another better allocation strategy.
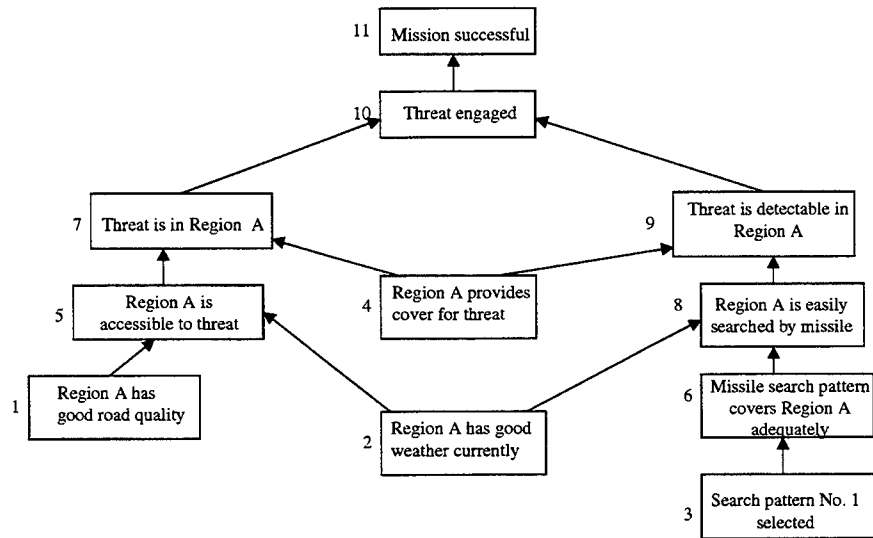
Figure 4.1-4. A Cruise Missile Search Pattern Decision Network

4.1.5 A Genetics Application

As a final illustration of Bayesian inferencing, a genetics application discussed by Jensen[?] is modelled and solved using DIRACNET. The problem may be simply stated as follows: *Given that a horse has manifested a genetic disease, determine which ancestors are carriers.* The DAG for this problem is shown in Figure 4.1-5, where letters A through K represent the assertions "horse A carries the gene," etc., and the arcs connect offspring with their parents. There, J and K have been crossed to produce a horse that has the disease which can be carried as a recessive gene. Since neither J nor K manifest the disease but have produced an offspring that does, then both J and K are known to carry the gene recessively. Given this information, a Dirac

network can be constructed and used to determine how likely it is that the genetic ancestors of J and K are carriers of the gene. Note that although the causal mechanism is in the direction from ancestor to child, the implication mechanism - as shown by the DAG - is in the opposite direction.

The Dirac truth state for this problem is

$$|\Psi\rangle = \hat{\partial}[\mathcal{R}] \, |\theta_A\rangle \, |\theta_B\rangle \, |\theta_C\rangle \, |\theta_D\rangle \, |\theta_E\rangle \, |\theta_F\rangle \, |\theta_G\rangle \, |\theta_H\rangle \, |\theta_I\rangle \, |\theta_J\rangle \,,$$

where $\hat{\partial}[\mathcal{R}]$ is the composite rotation operator ordering given by

$$\begin{aligned}
\hat{\partial}[\mathcal{R}] &= \hat{R}_{IE}\,(\alpha_{IE})\,\hat{R}_{IB}\,(\alpha_{IB})\,\hat{R}_{HD}\,(\alpha_{HD})\,\hat{R}_{HC}\,(\alpha_{HC})\,\hat{R}_{GC}\,(\alpha_{GC})\,\hat{R}_{GB}\,(\alpha_{GB}) \\
&\quad \hat{R}_{FB}\,(\alpha_{FB})\,\hat{R}_{FA}\,(\alpha_{FA})\,\hat{R}_{KI}\,(\alpha_{KI})\,\hat{R}_{KH}\,(\alpha_{KH})\,\hat{R}_{JG}\,(\alpha_{JG})\,\hat{R}_{JF}\,(\alpha_{JF})\,.
\end{aligned}$$

This system was initialized in DIRACNET with prior values

$$\theta_A = \theta_B = \theta_C = \theta_D = \theta_E = \theta_F = \theta_G = \theta_H = \theta_I = 89^0$$

and

$$\theta_J = \theta_K = 0^0.$$

These assignments reflect the facts that horses $A, B, C, D, E, F, G, H,$ and I are initially assumed to most probably not carry the gene and that horses J and K are known to carry the gene. All of the rotation angles used to propagate the influences used the same functional form given by

$$\alpha_{XY} = 0.5\theta'_X + 25^0,$$

where X and Y are variable (node) labels defined by $\mathcal{R}$ and θ'_X is the posterior angle in degrees for variable X (consequently, the order of rotation operator application is relevant).

A single propagation pass through the Dirac network in the direction of the associated DAG yielded results comparable to those given by Jensen[?]. These results are tabulated below:

X (node index)	θ'_X (degrees)	Posterior Probability That Assertion X is True
A	63	0.21
B	11	0.96
C	37	0.64
D	63	0.21
E	63	0.21
F	42	0.55
G	42	0.55
H	42	0.55
I	42	0.55
J	0	1.00
K	0	1.00

Additional useful information was provided by DIRACNET concerning the informational

uncertainty associated with the network (recall from above that a ratio of von Neumann entropies provides a measure of informational uncertainty). For this problem, DIRACNET yielded the following uncertainties for variable sets S, T, and U:

$$\Delta_S = 0.67, S = \{A, B, C, D, E\}$$

$$\Delta_T = 0.99, T = \{F, G, H, I\}$$

$$\Delta_U = 0.00, U = \{J, K\}.$$

These values indicate that the information associated with variable set: (1) U is completely certain (recall that $\Delta_U = 0.00$ is total certainty); (2) T is most uncertain (recall that $\Delta_T = 1.00$ is total uncertainty); and (3) S is "moderately" certain. Note that these uncertainties are consistent with the θ'_X angles and posterior probabilities in the last table. More importantly, observe that item (2) implies that additional information concerning the variables in set T would likely provide a "better" overall result.
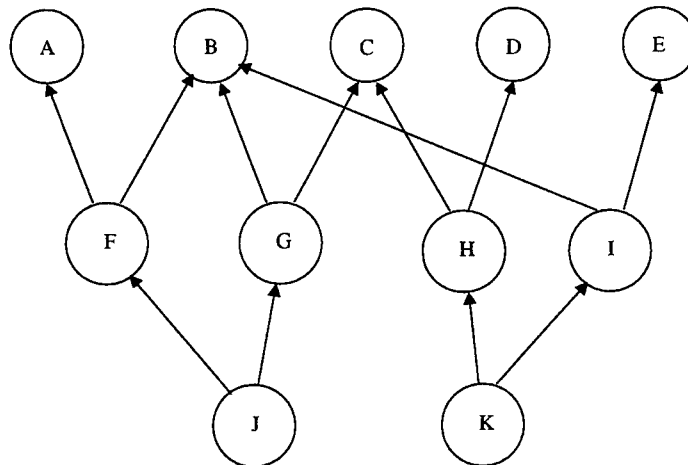
Figure 4.1-5. A Genetics Example

4.2 SYSTEMS ANALYSIS

In this section we illustrate how the Dirac algebra based methodology can be used to support system design trade-off studies. We will also develop and illustrate a perturbation theory which can be used for certain system sensitivity analyses.

4.2.1 A Simple System Design Trade-off Study

Consider the following simple model of a military weapon subsystem: A track file is prepared for a set of targets from the data gathered by a group of sensors. This track file is used to prepare a plan for a collection of weapons in order that they may engage the targets. Suppose we wish to use the Dirac algebraic method to contrast the performance of systems comprised of two such subsystems: (1) when the subsystems are permitted to operate autonomously; and (2) when the two subsystems are centrally coordinated. We will first apply this approach to describe general truth state vectors and associated performance parameters that we will use for the analysis of these systems. Then we shall discuss in closed form a special case where specific influences are assigned to the parent-child pairs associated with each system. Let us begin by examining a single subsystem.

4.2.1.1 The State Vector And Performance Parameters For A Single Subsystem

We model a single subsystem by using an influence network consisting of three variables which make assertions about the track file, the engagement plan, and the weapon-on-target engagements. The variable indices and associated assertions are given as follows:

Variable Index	Assertion
1	The track file quality for all targets is good.
2	A good engagement plan is generated and distributed to the weapons in time.
3	All targets are successfully engaged and neutralized.

The prior truth states for these variables are $|\theta_1\rangle$, $|\theta_2\rangle$ and $|\theta_3\rangle$, where the subscripts are the variable indices. Furthermore, assume that variable 1 exerts an α_{12} influence upon variable 2 and that variable 2 exerts an α_{23} influence upon variable 3. Then the truth state vector for this single system is

$$
\begin{aligned}
|\Phi\rangle &= \hat{\partial}[\mathcal{R}]\,|\theta_1\rangle\,|\theta_2\rangle\,|\theta_3\rangle \\
&= \hat{R}_{12}(\alpha_{12})\hat{R}_{23}(\alpha_{23})\,|\theta_1\rangle\,|\theta_2\rangle\,|\theta_3\rangle \\
&= |\theta_1\rangle\,|\theta_2 - \alpha_{12}\rangle\,|\theta_3 - \alpha_{23}\rangle,
\end{aligned}
$$

where $\hat{\partial}[\mathcal{R}]$ means that the specified order of application of the rotation operators is important (so that α_{12} and α_{23} are θ_1-dependent and θ_2-dependent rotation angles, respectively). When expanded in terms of truth basis vectors, this becomes

$$
\begin{aligned}
|\Phi\rangle = {}& \cos\theta_1 \cos(\theta_2 - \alpha_{12}) \cos(\theta_3 - \alpha_{23})\,|+++\rangle + \\
& \cos\theta_1 \cos(\theta_2 - \alpha_{12}) \sin(\theta_3 - \alpha_{23})\,|++-\rangle +
\end{aligned}
$$

$$\cos\theta_1 \sin(\theta_2 - \alpha_{12}) \cos(\theta_3 - \alpha_{23}) \left|+--+\right\rangle +$$
$$\sin\theta_1 \cos(\theta_2 - \alpha_{12}) \cos(\theta_3 - \alpha_{23}) \left|-++\right\rangle +$$
$$\cos\theta_1 \sin(\theta_2 - \alpha_{12}) \sin(\theta_3 - \alpha_{23}) \left|+--\right\rangle +$$
$$\sin\theta_1 \cos(\theta_2 - \alpha_{12}) \sin(\theta_3 - \alpha_{23}) \left|-+-\right\rangle +$$
$$\sin\theta_1 \sin(\theta_2 - \alpha_{12}) \cos(\theta_3 - \alpha_{23}) \left|--+\right\rangle +$$
$$\sin\theta_1 \sin(\theta_2 - \alpha_{12}) \sin(\theta_3 - \alpha_{23}) \left|---\right\rangle .$$

We will use the post-state expected value for the truth operator $\hat{\tau}_3$ for variable 3, i.e.

$$\langle \hat{\tau}_3 \rangle = \langle \theta_3 - \alpha_{23} | \hat{\tau}_3 | \theta_3 - \alpha_{23} \rangle ,$$

as a measure of this simple subsystem's performance. Thus, the closer the value of $\langle \hat{\tau}_3 \rangle$ is to 1 (-1) the more true (false) the assertion for variable 3 and the better (worse) the subsystem's performance. We shall also use the projection

$$
\begin{aligned}
prob(+++) &\equiv \langle \Phi | +++ \rangle \langle +++ | \Phi \rangle \\
&= \cos^2\theta_1 \cos^2(\theta_2 - \alpha_{12}) \cos^2(\theta_3 - \alpha_{23})
\end{aligned}
$$

and

$$\Delta_1 = \frac{E_1}{E_{1_{max}}},$$

where

$$E_1 = -\kappa \left(\cos^2\theta_1 \ln \cos^2\theta_1 + \sin^2\theta_1 \ln \sin^2\theta_1 \right)$$

and

$$E_{1_{max}} = \kappa \ln 2$$

as additional measures of performance. Clearly, $prob(+++)$ is the probability that the subsystem is in the most desirable truth basis state $|+++\rangle$ where all of the subsystem assertions are precisely true. If desired, other projection operators may be used to determine the probability that the system is in other truth basis states, e.g. it may be of interest to examine the sensitivity of the logical negation of assertion 3 (i.e., *it is false that all targets are successfully engaged and neutralized*) to changes in θ_1 by evaluating $prob(++-)$. The quantity Δ_1 provides a measure of the uncertainty - i.e. lack of information - associated with the truth state for variable 1, thereby quantifying the quality of the track files. Obviously, the entropy for any subset of variables can be used to provide other uncertainty measures. For example, it might be of interest to determine how $\langle \hat{\tau}_3 \rangle$ changes with variations in the quality of the engagement plan as quantified by $\Delta_2 = E_2/E_{2_{max}}$.

4.2.1.2 The Truth State Vector And Performance Parameters For Two Autonomous Subsystems

Consider now the truth state vector for two subsystems of the type described above which are operating autonomously. Let $A = \{1, 2, 3\}$ index the variables as defined above for the first

subsystem and let $B = \{4, 5, 6\}$ index an identical set of variables for the second subsystem. Also, let α_{12} and α_{23} be the influences in the first subsystem and let α_{45} and α_{56} be the influences in the second subsystem. The general truth state for a system comprised of two such autonomous subsystems is

$$
\begin{aligned}
|\Psi_a\rangle &= \hat{R}_{12}(\alpha_{12})\hat{R}_{23}(\alpha_{23})|\theta_1\rangle|\theta_2\rangle|\theta_3\rangle\,\hat{R}_{45}(\alpha_{45})\hat{R}_{56}(\alpha_{56})|\theta_4\rangle|\theta_5\rangle|\theta_6\rangle \\
&= |\Phi_{A_a}\rangle|\Phi_{B_a}\rangle .
\end{aligned}
$$

Here the subscript "a" indicates "for the autonomous system." Thus, the truth state vector for this "uncoupled" system is the tensor product of the truth state vectors for each subsystem. We note that this is a general result for a system comprised of a finite number of autonomous subsystems. In particular,

- If $\left\{ \left|\psi_j\right\rangle : 1 \leq j \leq n \right\}$ is a set of truth state vectors for n autonomous (non-interacting) subsystems which comprise a system, then $|\varphi_a\rangle = |\psi_1\rangle|\psi_2\rangle\cdots|\psi_n\rangle$ is the truth state vector for the system.

As before, we will use the post-state expected value for a system truth operator $\hat{T} = \frac{1}{2}(\hat{\tau}_3 + \hat{\tau}_6)$ for variables 3 and 6 to evaluate the system's performance:

$$
\left\langle \hat{T} \right\rangle_a = \frac{1}{2}\left(\langle \hat{\tau}_3 \rangle_a + \langle \hat{\tau}_6 \rangle_a \right).
$$

We shall also use $prob_a(+ + + + ++)$ and Δ_{14_a} as additional performance measures, where

$$
\begin{aligned}
prob(+ + + + ++)_a &= \langle \Psi_a | + + + + ++ \rangle \langle + + + + ++ | \Psi_a \rangle \\
&= \langle \Phi_{A_a} | + + + \rangle_{A_a A_a} \langle + + + | \Phi_{A_a} \rangle \cdot \\
&\quad \langle \Phi_{B_a} | + + + \rangle_{B_a B_a} \langle + + + | \Phi_{B_a} \rangle \\
&= prob_{A_a}(+ + +) prob_{B_a}(+ + +)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_{14_a} &\equiv \frac{E_{1_a} + E_{4_a}}{E_{1_{\max}} + E_{4_{\max}}} \\
&= \frac{E_{1_a} + E_{4_a}}{2\kappa \ln 2} \\
&= \frac{1}{2}\left(\frac{E_{1_a}}{\kappa \ln 2} + \frac{E_{4_a}}{\kappa \ln 2} \right) \\
&= \frac{1}{2}\left(\Delta_{1_a} + \Delta_{4a} \right).
\end{aligned}
$$

Observe that if these two subsystems are identical, i.e.,

$$
\theta_1 = \theta_4, \theta_2 = \theta_5, \theta_3 = \theta_6, \alpha_{12} = \alpha_{45}, \text{ and } \alpha_{23} = \alpha_{56},
$$

then

$$
\langle \hat{\tau}_3 \rangle_a = \langle \hat{\tau}_6 \rangle_a = \langle \hat{\tau}_3 \rangle \Rightarrow \left\langle \hat{T} \right\rangle_a = \langle \hat{\tau}_3 \rangle,
$$

$$prob_{A_a}(+++) = prob_{B_a}(+++) = prob(+++) \Rightarrow$$
$$prob_a(+++++) = [prob(+++)]^2,$$

and

$$\Delta_{1_a} = \Delta_{4_a} = \Delta_1 \Rightarrow \Delta_{14_a} = \Delta_1.$$

These properties have obvious generalizations for systems comprised of a finite number of identical autonomous subsystems. This suggests that certain aspects of the behavior of such systems may be ascertained from the study of a single subsystem of the system.

4.2.1.3 The Truth State Vector And Performance Parameters For Two Coordinated Subsystems

Now let the system be such that it consists of a pair of subsystems as just described with an additional variable indexed by 7 which coordinates the planning of the two subsystems via the influences α_{72} and α_{75}. Specifically, let variable 7 make the following assertion: *"The engagement plans for each subsystem are perfectly coordinated."*

For the purpose of illustration, we choose the rotation operator application order such that the general truth state vector for this system is

$$\begin{aligned}
|\Gamma_c\rangle &= \hat{R}_{12}(\alpha_{12})\hat{R}_{72}(\alpha_{72})\hat{R}_{23}(\alpha_{23})\,|\theta_1\rangle\,|\theta_2\rangle\,|\theta_3\rangle \cdot \\
&= \hat{R}_{45}(\alpha_{45})\hat{R}_{75}(\alpha_{75})\hat{R}_{56}(\alpha_{56})\,|\theta_4\rangle\,|\theta_5\rangle\,|\theta_6\rangle\,|\theta_7\rangle ,
\end{aligned}$$

where the subscript "c" indicates "for the coordinated system." The associated performance parameters of interest are

$$\left\langle \hat{T} \right\rangle_c = \frac{1}{2}\left(\langle\hat{\tau}_3\rangle_c + \langle\hat{\tau}_6\rangle_c \right),$$

$$prob_c(+++++++) = prob_{A_c}(+++)prob_{B_c}(+++)prob_7(+),$$

and

$$\Delta_{14_c} = \Delta_{14_a}.$$

This last relationship between uncertainty parameters results from the fact that decision variables 1 and 4 are independent in both the autonomous and coordinated systems. Also, observe that if $\alpha_{72} = \alpha_{75} = 0$, i.e., there is no coordination, then $\langle\theta_7|\,\Gamma_c\rangle = |\Psi_a\rangle$.

4.2.1.4 A Specific Instance

In this section, we will examine the performance of the two systems described above when the associated influences are defined as follows:

$$\alpha_{12} = -\theta_1,$$

$$\alpha_{23} = -\theta_2,$$

$$\alpha_{45} = -\theta_4,$$

$$\alpha_{56} = -\theta_5,$$

and

$$\alpha_{72} = \alpha_{75} = \beta(\frac{\pi}{2} - \theta_7).$$

Here it is assumed that the "coordinator" exerts the same influence over both engagement plans and the "strength" of the influence resides in the magnitude of the scaling parameter β. These influence assignments imply the following intuitive relationships:

- The quality of the engagement plans varies directly with the quality of the track files;

- The effectiveness of the engagements varies directly with the quality of the engagement plans; and

- The quality of the engagement plans varies directly with the quality of the coordination.

Using these assignments in the performance parameters for the autonomous system yields:

$$\left\langle \hat{T} \right\rangle_a = \frac{1}{2}[\cos^2(\theta_1 + \theta_2 + \theta_3) - \sin^2(\theta_1 + \theta_2 + \theta_3) + \cos^2(\theta_4 + \theta_5 + \theta_6) - \sin^2(\theta_4 + \theta_5 + \theta_6)],$$

$$prob_a(+ + + + + +) = \cos^2\theta_1 \cos^2(\theta_1 + \theta_2)\cos^2(\theta_1 + \theta_2 + \theta_3) \cdot \cos^2\theta_4 \cos^2(\theta_4 + \theta_5)\cos^2(\theta_4 + \theta_5 + \theta_6),$$

and

$$\Delta_{14_a} = -(\frac{1}{2\ln 2})(\cos^2\theta_1 \ln\cos^2\theta_1 + \sin^2\theta_1 \ln\sin^2\theta_1 + \cos^2\theta_4 \ln\cos^2\theta_4 + \sin^2\theta_4 \ln\sin^2\theta_4).$$

Similarly, for the coordinated system we obtain:

$$\left\langle \hat{T} \right\rangle_c = \frac{1}{2}\{\cos^2[\theta_1 + \theta_2 + \theta_3 - \beta(\frac{\pi}{2} - \theta_7)] - \sin^2[\theta_1 + \theta_2 + \theta_3 - \beta(\frac{\pi}{2} - \theta_7)] + \cos^2[\theta_4 + \theta_5 + \theta_6 - \beta(\frac{\pi}{2} - \theta_7)] - \sin^2[\theta_4 + \theta_5 + \theta_6 - \beta(\frac{\pi}{2} - \theta_7)]\},$$

$$prob_c(+ + + + + + +) = \cos^2\theta_1 \cos^2[\theta_1 + \theta_2 - \beta(\frac{\pi}{2} - \theta_7)] \cdot \cos^2[\theta_1 + \theta_2 + \theta_3 - \beta(\frac{\pi}{2} - \theta_7)] \cdot$$

$$\cos^2 \theta_4 \cos^2[\theta_4 + \theta_5 - \beta(\frac{\pi}{2} - \theta_7)] \cdot$$

$$\cos^2[\theta_4 + \theta_5 + \theta_6 - \beta(\frac{\pi}{2} - \theta_7)] \cdot$$

$$\cos^2 \theta_7,$$

and

$$\Delta_{14_c} = \Delta_{14_a}.$$

4.2.1.5 A Performance Comparison Of The Autonomous And Coordinated Systems Using The Truth Operator

Here we characterize the relative performances of the autonomous and coordinated systems using the ratio of $\langle \hat{T} \rangle_a$ and $\langle \hat{T} \rangle_c$. These values indicate how true - on average - the assertions are that are made by variables 3 and 6. Thus, we have a relative measure of how successful the weapon-on-target engagements are for each system.

To better understand the performance, let us parameterize the results in terms of sums of the truth angles using the following:

$$\gamma \equiv \theta_1 + \theta_2 + \theta_3$$

and

$$\eta \equiv \theta_4 + \theta_5 + \theta_6.$$

Also, for the coordinated system, let us assume that the assertion made by variable 7 is completely true so that $\theta_7 = 0$. Thus, the engagement plans for each of its subsystems are perfect and we say that *the coordinator is perfect*. We shall use β as the parameter which controls how influential this perfect coordination effort is upon each subsystem's separate engagement · plan. In particular, there is *no coordination* of subsystem plans when $\beta = 0$ and *complete coordination* when $\beta = 1$. Thus, $0 < \beta < 1$ reflects the level of coordination between these two extremes.

Using this parameterization yields the following ratio:

$$\frac{\langle \hat{T} \rangle_c}{\langle \hat{T} \rangle_a} = \{\cos^2[\gamma - \beta(\frac{\pi}{2})] - \sin^2[\gamma - \beta(\frac{\pi}{2})] +$$

$$\cos^2[\eta - \beta(\frac{\pi}{2})] - \sin^2[\eta - \beta(\frac{\pi}{2})]\}/$$

$$\{\cos^2 \gamma - \sin^2 \gamma + \cos^2 \eta - \sin^2 \eta\}.$$

Several observations can readily be made from this equation:

- When $\beta = 0$, then

$$\frac{\left\langle \hat{T} \right\rangle_c}{\left\langle \hat{T} \right\rangle_a} = 1.$$

This suggests the intuitively pleasing conclusion that *when coordination is non-existent the coordinated system's performance is identical to that of the autonomous system, even when the coordinator is perfect.*

- When

$$0 < \beta \le 1,$$

$$0 \le \gamma \le \beta(\frac{\pi}{2}),$$

and

$$0 \le \eta \le \beta(\frac{\pi}{2}),$$

then, from the door-stop rule, we have

$$\frac{\left\langle \hat{T} \right\rangle_c}{\left\langle \hat{T} \right\rangle_a} = \frac{2}{\cos^2\gamma - \sin^2\gamma + \cos^2\eta - \sin^2\eta}$$

$$> 1 \text{ when } 0 < \gamma, \eta < \frac{\pi}{4}$$

$$< 1 \text{ when } \frac{\pi}{4} < \gamma, \eta \le \frac{\pi}{2}.$$

Thus, we conclude that the performance of the coordinated system is superior to that of the autonomous system for all $0 \le \beta < \frac{1}{2}$ since

$$0 < \gamma, \eta < \beta(\frac{\pi}{2}) < \frac{\pi}{4} \Rightarrow \frac{\left\langle \hat{T} \right\rangle_c}{\left\langle \hat{T} \right\rangle_a} > 1.$$

This suggests that *"small" to "moderate" levels of subsystem coordination enhance the coordinated system's performance over that for the autonomous system when the imperfections in the two subsystems are also at "small" to "moderate" levels.* We also conclude that the performance of the coordinated system is exceptionally superior to that of the autonomous system for all $\frac{1}{2} < \beta \le 1$ since

$$\frac{\pi}{4} < \gamma, \eta \le \beta(\frac{\pi}{2}) \le \frac{\pi}{2} \Rightarrow$$

$$\left\langle \hat{T} \right\rangle_a < 0 \text{ and } \left\langle \hat{T} \right\rangle_c = 1 \Rightarrow$$

$$\frac{\left\langle \hat{T} \right\rangle_c}{\left\langle \hat{T} \right\rangle_a} < 0.$$

Note that in this γ, η regime the performance of the autonomous system is poor since

$\left\langle \hat{T} \right\rangle_a < 0$ while that of the coordinated system is still perfect since $\left\langle \hat{T} \right\rangle_c = 1$. This suggests that *"significant" levels of subsystem coordination can be used to enhance the coordinated system's performance over that of the autonomous system when the subsystem imperfections are also "significant" - provided that the coordinator is perfect.*

4.2.1.6 A Performance Comparison Of The Autonomous And Coordinated Systems Using Projections

Clearly, $|{+}{+}{+}{+}{+}{+}\rangle$ and $|{+}{+}{+}{+}{+}{+}{+}\rangle$ are the most desirable of all possible truth basis states for the autonomous and coordinated systems, respectively. We shall refer to these states as the *exalted states* for these systems. When these systems exist only (i.e., with unit probability) in these exalted states they are functioning perfectly. However, when imperfections exist in these systems (i.e., when variable truth states are superpositions of $|+\rangle$ and $|-\rangle$), then the truth state vectors for these systems become superpositions of other truth basis states as well. In this case, these systems possess non-vanishing probabilities that they can exist in other truth basis states in addition to the exalted states. Thus, *the more probable it is that these systems are in exalted states, the better their performance* is expected to be.

Here, we will use the expressions for $prob_a({+}{+}{+}{+}{+}{+})$ and $prob_c({+}{+}{+}{+}{+}{+}{+})$ developed above from the associated projections to evaluate the performance of these systems from this perspective. Let

$$\begin{aligned}
\zeta &= \theta_1 + \theta_2, \\
\vartheta &= \zeta + \theta_3, \\
\lambda &= \theta_4 + \theta_5, \\
\kappa &= \lambda + \theta_6,
\end{aligned}$$

and consider the ratio

$$\begin{aligned}
r &\equiv \frac{prob_c({+}{+}{+}{+}{+}{+}{+})}{prob_a({+}{+}{+}{+}{+}{+})} \\
&= \{\cos^2[\zeta - \beta(\frac{\pi}{2} - \theta_7)]\cos^2[\vartheta - \beta(\frac{\pi}{2} - \theta_7)] \cdot \\
&\quad \cos^2[\lambda - \beta(\frac{\pi}{2} - \theta_7)]\cos^2[\kappa - \beta(\frac{\pi}{2} - \theta_7)]\cos^2\theta_7\} / \\
&\quad \{\cos^2\zeta\cos^2\vartheta\cos^2\lambda\cos^2\kappa\}.
\end{aligned}$$

The following observations are worthy of note:

- If $\beta = 0$ (i.e., when there is no coordination of subsystem plans), then we conclude

$$0 \leq r = \cos^2\theta_7 \leq 1.$$

As expected, this suggests that *without coordination, it is more probable for the autonomous system to exist in its exalted state than it is for the coordinated system to exist in its exalted state. However, if the coordinator is perfect (i.e., $\theta_7 = 0$), then these probabilities will always be equal.*

- When $0 < \beta < 1$, $\theta_7 = 0$ (i.e., the coordinator is perfect),

$$0 \leq \vartheta \leq \beta(\frac{\pi}{2}),$$

and

$$0 \leq \kappa \leq \beta(\frac{\pi}{2}),$$

then, from the door-stop rule, we have

$$
\begin{aligned}
r &= \frac{1}{\cos^2 \zeta \cos^2 \vartheta \cos^2 \lambda \cos^2 \kappa} \\
&> 1 \text{ when } 0 < \zeta, \vartheta, \lambda, \kappa < \frac{\pi}{2}.
\end{aligned}
$$

(Clearly, when $0 \leq \vartheta, \kappa \leq \beta(\frac{\pi}{2})$, then $0 \leq \zeta, \lambda \leq \beta(\frac{\pi}{2})$ because $0 \leq \theta_3, \theta_6$). Thus, we conclude that for all levels of coordination between the extremes of no coordination and complete coordination, the probability for the coordinated system to be in its exalted state is greater than that for the autonomous system. This suggests that *"very small" to "nearly complete" levels of coordination can be used to increase the probability of existence in the exalted state for the coordinated system relative to that for the autonomous system when the subsystem imperfections are "small" to "moderate" in magnitude and the coordination is perfect.*

4.2.1.7 Changes In The Performance Of The Qutonomous And Coordinated Systems Due To Variations In The Target Track File Uncertainty

As a final illustration, we will examine one of the more obvious relationships between entropy and system performance. In this model, the uncertainty $\Delta_{14_a} = \Delta_{14_c} \equiv \Delta_{14}$ quantifies the quality of the target track files for the autonomous and coordinated systems. Let us assume that the prior states for variables 2, 3, 5, and 6 are perfect (i.e., $\theta_2 = \theta_3 = \theta_5 = \theta_6 = 0$) and that each subsystem's track file is of identical quality, i.e. $\theta_1 = \theta_4 = \varphi$. Furthermore, assume that for the coordinated system, the quality of the "coordinator" is degraded by track file quality so that $\theta_7 = \varphi$. Then,

$$
\begin{aligned}
\left\langle \hat{T} \right\rangle_a &= \cos^2 \varphi - \sin^2 \varphi \\
&= \cos 2\varphi,
\end{aligned}
$$

$$
\left\langle \hat{T} \right\rangle_c = \cos^2[(1+\beta)\varphi - \beta(\frac{\pi}{2})] - \sin^2[(1+\beta)\varphi - \beta(\frac{\pi}{2})]
$$

$$= \cos[2(1+\beta)\varphi - \beta\pi],$$

and

$$\Delta_{14} = -(\frac{1}{\ln 2})(\cos^2\varphi \ln\cos^2\varphi + \sin^2\varphi \ln\sin^2\varphi).$$

Hence, as $\varphi \to \frac{\pi}{2}$, then $\left\langle \hat{T} \right\rangle_a \to -1$, $\left\langle \hat{T} \right\rangle_c \to -1$, and $\Delta_{14} \to 1$ as $\varphi \to \frac{\pi}{4}$ (the track file becomes more uncertain) and $\Delta_{14} \to 0$ as $\varphi \to \frac{\pi}{2}$ (the track file becomes more certainly of poor quality). This suggests the obvious conclusion that *decreasing track file quality generally degrades the performance of both the autonomous and coordinated systems.*

Now observe that $\left\langle \hat{T} \right\rangle_a$ and $\left\langle \hat{T} \right\rangle_c$ have periods π and $\left(\frac{\pi}{1+\beta}\right)$, respectively, and $\left\langle \hat{T} \right\rangle_c$ is phase shifted (towards increasing φ) relative to $\left\langle \hat{T} \right\rangle_a$ by $\left(\frac{\beta}{1+\beta}\right)\pi$. This means that for small β and φ values, the track file quality is still very good and $\left\langle \hat{T} \right\rangle_c > \left\langle \hat{T} \right\rangle_a$. This suggests that *for "small" decreases in track file quality "small" levels of coordination may enhance the performance of the coordinated system over that of the autonomous system.* However, for small φ there is a (small) β for which $\left\langle \hat{T} \right\rangle_a > \left\langle \hat{T} \right\rangle_c$ (e.g. for $\varphi = 0$, $\left\langle \hat{T} \right\rangle_a = +1$ and $\left\langle \hat{T} \right\rangle_c = \cos[-\beta\pi] < +1$). This suggests that *"over-coordination" when all subsystems have a high quality track file can decrease the performance of the coordinated system relative to that of the autonomous system.* Similarly, for a moderate φ value, there is a moderate $\beta = \beta_0$ for which $\left\langle \hat{T} \right\rangle_a > \left\langle \hat{T} \right\rangle_c$ for $\beta < \beta_0$ and $\left\langle \hat{T} \right\rangle_c \geq \left\langle \hat{T} \right\rangle_a$ for $\beta \geq \beta_0$ (e.g. $\varphi = 15^0 \Rightarrow \Delta_{14} = .3546$, $\beta_0 = 0.4$). We can conclude from this that *moderate coordination of a system using a track file of moderately degraded quality can increase the system's performance.* Although care must be taken when drawing conclusions from such an analysis, it is easy to see that the period and phase difference between $\left\langle \hat{T} \right\rangle_a$ and $\left\langle \hat{T} \right\rangle_c$ will produce conditions which yield the reasonable conclusion that *"over-coordination" based upon low quality track file information can yield a "precipitous" reduction in the performance of the coordinated system.*

4.2.2 System Sensitivity Analyses

4.2.2.1 A Static Perturbation Theory For Truth States

In this section we will develop a *first-order static* (time-independent) perturbation theory for truth states. Although this theory is general, its primary utility is for the estimation of the changes in the value of the truth state and expected truth values for nodes induced by small changes in variable truth angles and influence angles for systems which have *constant influence angles* or influence angles with *weak functional dependencies* upon parent truth angles.

Consider first the effect of a small rotation $\delta\theta_j$ upon a truth state $|\theta_j\rangle$:

$$\begin{aligned} \hat{R}(\delta\theta_j)|\theta_j\rangle &= |\theta_j - \delta\theta_j\rangle \\ &= e^{i \cdot \delta\theta_j \cdot \hat{\sigma}_j}|\theta_j\rangle \\ &\approx (\hat{1}_j + i \cdot \delta\theta_j \cdot \hat{\sigma}_j)|\theta_j\rangle \end{aligned}$$

$$\approx \ |\theta_j\rangle + i \cdot \delta\theta_j \cdot \hat{\sigma}_j |\theta_j\rangle .$$

Here we use a "generic" rotation operator $\hat{R}(\delta\theta_j)$ to initiate our development. Now let

$$|\Psi\rangle = |\theta_1\rangle |\theta_2\rangle \cdots |\theta_n\rangle \qquad (4\text{-}1)$$

so that in general

$$
\begin{aligned}
|\Psi + \delta\Psi\rangle &= |\theta_1 - \delta\theta_1\rangle |\theta_2 - \delta\theta_2\rangle \cdots |\theta_n - \delta\theta_n\rangle \\
&\approx (|\theta_1\rangle + i\delta\theta_1\hat{\sigma}_1 |\theta_1\rangle)(|\theta_2\rangle + i\delta\theta_2\hat{\sigma}_2 |\theta_2\rangle) \cdots \\
&\quad (|\theta_n\rangle + i\delta\theta_n\hat{\sigma}_n |\theta_n\rangle).
\end{aligned}
$$

To first-order in $\delta\theta_j$ this becomes

$$
\begin{aligned}
|\Psi + \delta\Psi\rangle &\approx |\theta_1\rangle |\theta_2\rangle \cdots |\theta_n\rangle + i\sum_{j=1}^{n} \delta\theta_j\hat{\sigma}_j |\theta_j\rangle \left|\phi_j\right\rangle \\
&\approx |\Psi\rangle + i\sum_{j=1}^{n} \delta\theta_j\hat{\sigma}_j |\theta_j\rangle \left|\phi_j\right\rangle ,
\end{aligned}
$$

or

$$
\begin{aligned}
|\delta\Psi\rangle &\equiv |\Psi + \delta\Psi\rangle - |\Psi\rangle \qquad (4\text{-}2) \\
&\approx i\sum_{j=1}^{n} \delta\theta_j\hat{\sigma}_j |\theta_j\rangle \left|\phi_j\right\rangle ,
\end{aligned}
$$

where $|\delta\Psi\rangle$ is the first-order change in the truth state $|\Psi\rangle$ due to the small rotations $\delta\theta_j$ and

$$\left|\phi_j\right\rangle = |\theta_1\rangle |\theta_2\rangle \cdots |\theta_{j-1}\rangle |\theta_{j+1}\rangle \cdots |\theta_n\rangle .$$

Note that,in general, each child $\delta\theta_j$ can be dependent upon the associated perturbations in its parent variables.

Let us now examine how a small change in a truth state effects the expected value for the associated truth operator. Since $\hat{\sigma}_j$ is Hermitean, then to first-order in $\delta\theta_j$

$$
\begin{aligned}
\langle\theta_j - \delta\theta_j| \hat{\tau}_j |\theta_j - \delta\theta_j\rangle &\approx ((\langle\theta_j| - i \cdot \delta\theta_j \cdot \langle\theta_j| \hat{\sigma}_j)\hat{\tau}_j(|\theta_j\rangle + i \cdot \delta\theta_j \cdot \hat{\sigma}_j |\theta_j\rangle)) \\
&\approx \langle\theta_j| \hat{\tau}_j |\theta_j\rangle + i \cdot \delta\theta_j \cdot \langle\theta_j| [\hat{\tau}_j, \hat{\sigma}_j] |\theta_j\rangle \\
&\approx \langle\hat{\tau}_j\rangle + i \cdot \delta\theta_j \cdot \langle[\hat{\tau}_j, \hat{\sigma}_j]\rangle ,
\end{aligned}
$$

or

$$
\begin{aligned}
\delta\langle\hat{\tau}_j\rangle &\equiv \langle\theta_j - \delta\theta_j| \hat{\tau}_j |\theta_j - \delta\theta_j\rangle - \langle\hat{\tau}_j\rangle \qquad (4\text{-}3) \\
&\approx i \cdot \delta\theta_j \cdot \langle[\hat{\tau}_j, \hat{\sigma}_j]\rangle .
\end{aligned}
$$

Here, $\delta\langle\hat{\tau}_j\rangle$ is the (first-order) change in the expected truth value for variable j due to the small

rotation $\delta\theta_j$. Recall from lemma 13 that $[\hat{\tau}_j, \hat{\sigma}_j] = \hat{\pi}_j$, where $\hat{\pi}_j$ is the permutation operator for variable j. Using this result in Equation(4-3) yields

$$\delta\left\langle\hat{\tau}_j\right\rangle \approx 2\cdot\delta\theta_j\cdot\left\langle\hat{\pi}_j\right\rangle.$$

Application of lemma 14 to this expression yields

$$\begin{aligned}
\delta\left\langle\hat{\tau}_j\right\rangle &\approx 2\cdot\delta\theta_j\cdot\Delta\tau_j \\
&\approx 4\cdot\delta\theta_j\cdot\sin\theta_j\cos\theta_j \\
&\approx 2\cdot\delta\theta_j\cdot\sin 2\theta_j.
\end{aligned}$$

Observe from this that $\delta\left\langle\hat{\tau}_j\right\rangle$ is a weighted truth uncertainty for the jth variable.

Let $\hat{T}$ be the generalized truth operator for a system with truth state vector $|\Psi\rangle$ given by Equation(4-1). Then to first-order

$$\begin{aligned}
\langle\Psi+\delta\Psi|\hat{T}|\Psi+\delta\Psi\rangle &= ((\langle\Psi|+\langle\delta\Psi|)\hat{T}(|\Psi\rangle+|\delta\Psi\rangle)) \\
&\approx \langle\Psi|\hat{T}|\Psi\rangle+\langle\Psi|\hat{T}|\delta\Psi\rangle+\langle\delta\Psi|\hat{T}|\Psi\rangle
\end{aligned}$$

so that

$$\begin{aligned}
\delta\langle\Psi|\hat{T}|\Psi\rangle &\equiv \langle\Psi+\delta\Psi|\hat{T}|\Psi+\delta\Psi\rangle-\langle\Psi|\hat{T}|\Psi\rangle \\
&\approx \langle\Psi|\hat{T}|\delta\Psi\rangle+\langle\delta\Psi|\hat{T}|\Psi\rangle,
\end{aligned}$$

where $\delta\langle\Psi|\hat{T}|\Psi\rangle$ is the change in the expected value for the general truth operator produced by a change $|\delta\Psi\rangle$ in the truth state $|\Psi\rangle$. To evaluate each term on the right hand side of the last expression, we use Equations(4-1) and (4-2) to obtain

$$\begin{aligned}
\langle\Psi|\hat{T}|\delta\Psi\rangle+\langle\delta\Psi|\hat{T}|\Psi\rangle &\approx i\sum_{j=1}^{n}\delta\theta_j\left\langle\phi_j\right|\left\langle\theta_j\right|\left[\hat{T},\hat{\sigma}_j\right]|\theta_j\rangle\left|\phi_j\right\rangle \\
&\approx \frac{i}{n}\sum_{j=1}^{n}\delta\theta_j\left\langle[\hat{\tau}_j,\hat{\sigma}_j]\right\rangle \\
&\approx \frac{2}{n}\sum_{j=1}^{n}\delta\theta_j\left\langle\hat{\pi}_j\right\rangle \\
&\approx \frac{2}{n}\sum_{j=1}^{n}\delta\theta_j\cdot\Delta\tau_j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta\langle\Psi|\hat{T}|\Psi\rangle &\approx \frac{2}{n}\sum_{j=1}^{n}\delta\theta_j\cdot\Delta\tau_j \\
&\approx \frac{2}{n}\sum_{j=1}^{n}\delta\theta_j\sin 2\theta_j.
\end{aligned}$$

Note from this that $\delta \langle \Psi | \hat{T} | \Psi \rangle$ is a weighted sum of logical variable truth uncertainties.

4.2.2.2   An Illustration: The Cruise Missile Search And Destroy Problem

Let us revisit the previous example which showed how the Dirac method can be applied to the cruise missile search and destroy allocation problem. Recall that although the influence angles for this system are functionally dependent upon the parent variable truth angles, this dependence appears in the argument of the exponential of the associated sigmoid scaling factors γ_j. These scaling factors are consequently fairly insensitive to small changes in parent truth angles and can be treated as strict functions of the associated unperturbed parent truth angles. Thus, perturbation theory can be easily employed in a straightforward manner to this system for the purpose of sensitivity analysis.

Suppose that one wishes to know how sensitive the overall quality of the system defined by Figure 4.1-4 is to *small* changes in the influence of "cover for threat" (variable 4). This can be estimated by assuming that the influence angles α_{47} and α_{49} are changed by the amounts $\delta\theta_7$ radians and $\delta\theta_9$ radians, respectively. Then, using the posterior values for θ_7 and θ_9, we find that

$$
\begin{aligned}
\delta \langle \hat{\tau}_7 \rangle &\approx 2 \cdot \delta\theta_7 \cdot \sin 2\theta_7 \\
&\approx 2 \cdot \delta\theta_7 \cdot \sin 52^0 \\
&\approx 1.576 \cdot \delta\theta_7,
\end{aligned}
$$

$$
\begin{aligned}
\delta \langle \hat{\tau}_9 \rangle &\approx 2 \cdot \delta\theta_9 \cdot \sin 2\theta_9 \\
&\approx 2 \cdot \delta\theta_9 \cdot \sin 90^0 \\
&\approx 2 \cdot \delta\theta_9,
\end{aligned}
$$

and

$$
\begin{aligned}
\delta \langle \Psi | \hat{T} | \Psi \rangle &\approx \frac{2}{11} \left[\delta\theta_7 \cdot \sin 52^0 + \delta\theta_9 \cdot \sin 90^0 \right] \\
&\approx \frac{2}{11} \left[0.788 \cdot \delta\theta_7 + \delta\theta_9 \right].
\end{aligned}
$$

Thus, positive changes in the influence of the "cover for threat" variable upon variables 7 ("threat in region") and 9 ("threat is detectable") will increase the expected truth values for variables 7 and 9, as well as for the entire system. For example, if $\delta\theta_7 = \delta\theta_9 = 5^0 \approx .0873$ radians, then $\delta \langle \hat{\tau}_7 \rangle \approx +0.138$, $\delta\hat{\tau}_9 \approx +0.175$, and $\delta \langle \Psi | \hat{T} | \Psi \rangle \approx +0.028$.

4.2.3   Data Representation and Analysis

In this section we will illustrate the utility of the Dirac method for data representation and analysis by applying it to two suggestive examples: (i) a rotational symmetry analysis of a simple digital image; and (ii) creating a fused observational error profile using an ε state.

4.2.3.1 Finding Rotational Symmetries In A Simple Digital Image

Consider the following square 8×8 pixel digital image:

$$
\begin{array}{|cccccccc|}
\hline
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

(4-4)

and partition it into a 2×2 quartered image such that each 4 pixel $\times 4$ pixel quarter image is ordered with indices $1, 2, 3, 4$ as follows:

$$
\begin{array}{|cc|}
\hline
1 & 2 \\
4 & 3 \\
\hline
\end{array}
$$

(4-5)

Here

$$
\boxed{1} \equiv
\begin{array}{|cccc|}
\hline
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\hline
\end{array}
$$

(4-6)

$$
\boxed{2} = \boxed{4} \equiv
\begin{array}{|cccc|}
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

and

$$
\boxed{3} \equiv
\begin{array}{|cccc|}
\hline
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

and 1 (on) and 0 (off) signify binary pixel values.

For the jth quarter image, let the probability $p_{j,1(0)}$ that a pixel is on (off) in the jth quarter image be $\frac{1}{16} \times$ [the number of 1's (0') in the jth quarter image] and define a truth state $|j\rangle$ for the jth quarter image as

$$
|j\rangle = \sqrt{p_{j,1}} \, |+\rangle_j + \sqrt{p_{j,0}} \, |-\rangle_j \, , j \in \{1, 2, 3, 4\},
$$

where $|j\rangle$ represents the truth state for the assertion "all pixels are on in the jth quarter image."

Using these definitions, we have

$$|1\rangle = \frac{\sqrt{2}}{2}[|+\rangle_1 + |-\rangle_1],$$

$$|2\rangle = |-\rangle_2,$$

$$|3\rangle = \frac{\sqrt{2}}{2}[|+\rangle_3 + |-\rangle_3],$$

and

$$|4\rangle = |-\rangle_4,$$

so that a truth state $|\Psi\rangle$ for the quarted image can be represented as the tensor product

$$
\begin{aligned}
|\Psi\rangle &= |1\rangle\,|2\rangle\,|3\rangle\,|4\rangle \\
&= \frac{1}{2}[|+--+-\rangle + |+----\rangle + \\
&\quad |--+-\rangle + |----\rangle].
\end{aligned}
$$

For the purpose of our simple symmetry analysis, suppose we ask: "*does the original image (4-4) have a $\frac{n\pi}{2}, n \in \{1,2,3\}$, (clockwise) rotational symmetry about its center when viewed as a (coarser grained) quartered image?*" Observe that the nth such rotation of the quartered image with order defined by Figure (4-5) is permutation π_n of the order indices in Figure (4-5). In particular, application of π_n to Figure (4-5) produces

$$
\begin{array}{|cc|}
\hline
4 & 1 \\
3 & 2 \\
\hline
\end{array}, \text{ when } n = 1,
$$

$$
\begin{array}{|cc|}
\hline
3 & 4 \\
2 & 1 \\
\hline
\end{array}, \text{ when } n = 2,
$$

and

$$
\begin{array}{|cc|}
\hline
2 & 3 \\
1 & 4 \\
\hline
\end{array}, \text{ when } n = 3.
$$

Consequently, each rotation written in Cayley cycle notation is

$$\pi_1 = (1432),$$

$$\pi_2 = (13)(24),$$

and

$$\pi_3 = (1234).$$

These permutations can be defined as operators $\hat{\pi}_n$ (*not to be confused with the permutation operator $\hat{\pi}_j$ for the jth truth state*) which act upon $|\Psi\rangle$ by acting upon each of its four basis ket vectors by permuting the order of their $\pm$'s (e.g., $\hat{\pi}_1\,|+--+-\rangle = |-+-+-\rangle$, etc) so that $\hat{\pi}_n\,|\Psi\rangle \equiv |\Psi_n\rangle$ is the truth state for the associated rotated quartered image. Thus, the expected

$n = 3$,

$$\hat{\pi}_3 \ket{\Psi} = \frac{1}{2}[\ket{-+-+} + \ket{-+--} + \ket{---+} + \ket{----}]$$

and

$$\langle \hat{\pi}_3 \rangle = \frac{1}{4} \langle ---- \ket{----} \rangle$$
$$= \frac{1}{4},$$

i.e. $\ket{\Psi}$ and $\ket{\Psi_3}$ have only the basis ket $\ket{----}$ in common.

Since $\langle \hat{\pi}_2 \rangle = 1$, we conclude that since the quartered image exhibits a perfect $\langle \Psi | \Psi_2 \rangle$ correlation, then the *image (4-4) possesses a coarse grained $\hat{\pi}_2$ (i.e. a 180^0 rotational) symmetry*. However, it is easy to see that the original image (4-4) does not possess an exact (non-coarse grained) 180^0 rotational symmetry because the 1st 3rd quarters in the rotated original image are

1	1	0	0
1	1	0	0
0	0	1	1
0	0	1	1

and

0	0	1	1
0	0	1	1
1	1	0	0
1	1	0	0

which are - themselves - 180^0 rotations of the 1st and 3rd quarters of the unrotated original image.

We can use the same quartering approach to find coarse grained symmetries in each quarter of image (4-4), as well as in each quarter of each of these quarters. Let's use the 1st quarter of image (4-4) given by subimage (4-6) to illustrate this. Using the same ordering scheme as (4-5) we have for this subimage

$$\boxed{1} = \boxed{3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\boxed{2} = \boxed{4} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Then the truth state $\ket{\Phi}$ for this subimage is the single ket

$$\ket{\Phi} = \ket{-+-+}$$

so that

$$\begin{aligned}
\langle \hat{\pi}_1 \rangle &= \langle \Phi | \, \hat{\pi}_1 \, | \Phi \rangle \\
&= \langle - + - + | \, \hat{\pi}_1 \, | - + - + \rangle \\
&= \langle - + - + | \, + - + - \rangle \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle \hat{\pi}_2 \rangle &= \langle \Phi | \, \hat{\pi}_2 \, | \Phi \rangle \\
&= \langle - + - + | \, \hat{\pi}_2 \, | - + - + \rangle \\
&= \langle - + - + | \, - + - + \rangle \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\langle \hat{\pi}_3 \rangle &= \langle \Phi | \, \hat{\pi}_3 \, | \Phi \rangle \\
&= \langle - + - + | \, \hat{\pi}_3 \, | - + - + \rangle \\
&= \langle - + - + | \, + - + - \rangle \\
&= 0.
\end{aligned}$$

We conclude from this that *the 1st quarter of image (4-4) also exhibits a coarse-grained $\hat{\pi}_2$ symmetry.* By observation, it is easy to see that the 3rd quarter of this image also exhibits a coarse-grained $\hat{\pi}_2$ symmetry and that the 2nd and 4th quarters exhibit all coarse-grained $\hat{\pi}_1$, $\hat{\pi}_2$, and $\hat{\pi}_3$ symmetries.

Let us now examine the non-coarse-grained pixel level symmetries for each quarter of the 1st subimage quarter. Using again the order defined by (4-5), it is easy to see that the truth states for each subimage quarter $1-4$ are

$$\begin{aligned}
|1\rangle &= |----\rangle, \\
|2\rangle &= |++++\rangle, \\
|3\rangle &= |----\rangle,
\end{aligned}$$

and

$$|4\rangle = |++++\rangle.$$

It is easy to see that for this subimage

$$\langle k | \, \hat{\pi}_j \, | k \rangle = 1, j \in \{1, 2, 3\}, k \in \{1, 2, 3, 4\}.$$

In a similar manner, it can be readily shown that the same value for $\langle \hat{\pi}_j \rangle, j \in \{1, 2, 3\}$, is obtained for each quarter of the remaining three subimage quarters of (4-4). We conclude from this that *each quarter of every subimage quarter of (4-4) exhibits pixel level $\hat{\pi}_j, j \in \{1, 2, 3\}$, symmetries.*

These results are summarized in the *rotational symmetry quad-tree* of Figure 4.2-1 which represents the $\frac{n\pi}{2}, n \in \{1,2,3\}$, rotational symmetries exhibited by image (4-4) when examined from different levels of image refinement. Here, the root vertex of the tree represents the original image and the label "2" indicates that when - quartered and rotated about its center - it exhibits an associated coarse $\hat{\pi}_2$ symmetry. The leaf vertices in the tree represent the finest pixel level quartering resolution possible for the image - each vertex represents such a quartering - and the labels $1, 2, 3$ on each of them indicates that each such quartering exhibits all $\hat{\pi}_1, \hat{\pi}_2$, and $\hat{\pi}_3$ (exact) symmetries when rotated about their centers. The four middle layer vertices in the tree represent quarterings of each quarter in the original image and their labels correspond to their associated rotational symmetries. Edges in the tree connect quarterings of quarters and for each such quartering the left-most vertex represents quarter 1 as defined by (4-5), the next left-most represents quarter 2, etc. An analysis of this symmetry quad-tree suggests that the image consists of diagonal arrangements of squares.
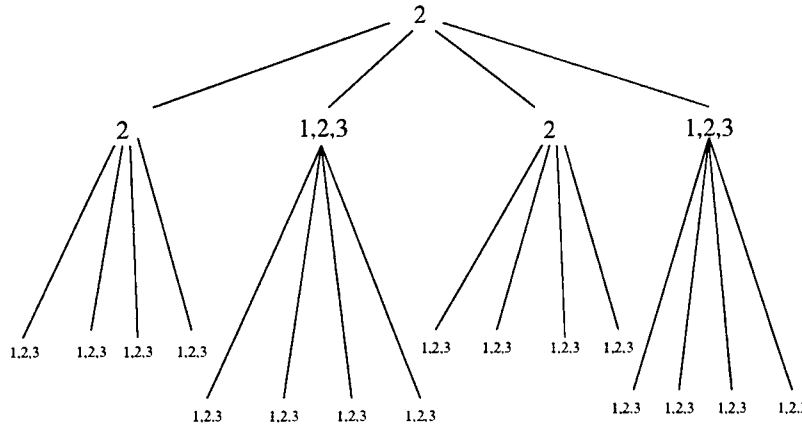

Figure 4.2-1. A Rotational Symmetry Quad-Tree

4.2.3.2 Observational Error Fusion Using An ε State

Let M be a surveillance instrument which can measure two signatures s_1 or s_2 that can be used to distinguish two categories of flying aircraft. The measurement errors associated with each signature are normally distributed with mean values s_1 and s_2 and standard deviations σ_1 and σ_2, respectively, such that $s_1 < s_2$, $\sigma_1 \approx \sigma_2$, and

$$|s_1 - s_2| < \min\{\sigma_1, \sigma_2\}.$$

This condition obviously indicates that serious aircraft classification difficulties can arise when measurements from M are used for this purpose. Fortunately, these signature measurements are sensitive to the elevation θ at which they are performed. Only the s_2 signature can be

observed when the aircraft is observed at M's zenith and only the s_1 signature can be observed when the aircraft is flying at very low elevations near M's local horizon. It is also known that s_1 signatures are increasingly more likely to be observed than s_2 measurements as $\theta \to 0^0$ and that it is equally likely for s_1 or s_2 to be observed if the measurement is made at local elevations near 45^0.

This behavior of the observational errors for M can be modelled using the system ket

$$
\begin{aligned}
|\Psi\rangle &= \hat{R}_{13}(\alpha_{13})\hat{R}_{23}(\alpha_{23})\,|\theta_1\rangle\,|\theta_2\rangle\,|\theta_3; \psi_3^{s_1,s_2}\rangle \tag{4-7}\\
&= |\theta_1\rangle\,|\theta_2\rangle\,|\theta_3 - (\alpha_{13} + \alpha_{23})\rangle,
\end{aligned}
$$

where θ_1 is the current observational elevation, θ_2 is the elevation of the previous observation, $\alpha_{13} = -\theta_1$, $\alpha_{23} = \theta_2$, and

$$
|\theta_3; \psi_3^{s_1,s_2}\rangle = \cos\theta_3\,|+\rangle_3\,|\psi_3^{s_1}\rangle + \sin\theta_3\,|-\rangle_3\,|\psi_3^{s_2}\rangle.
$$

The prior values for the truth angles are $\theta_1 = \theta_2 = \theta_3 = 0$ and the observation elevations θ_1 and θ_2 are assumed to be available and frequently updated. The associated error profile (i.e. the probability distribution for measuring s_1 or s_2) is

$$
|\langle\varphi_3\,|\Psi\rangle|^2 = \cos^2\beta\,|\langle\varphi_3\,|\psi_3^{s_1}\rangle|^2 + \sin^2\beta\,|\langle\varphi_3\,|\psi_3^{s_2}\rangle|^2,
$$

where $\beta = \theta_3 + (\theta_1 - \theta_2)$ and $|\langle\varphi_3\,|\psi_3^{s_1}\rangle|^2$ and $|\langle\varphi_3\,|\psi_3^{s_2}\rangle|^2$ are the Gaussian observational error profiles for s_1 and s_2, respectively.

To see how $|\Psi\rangle$ changes with elevation angle, let us iterate it through several observational cycles. Denoting the state and elevations for the kth iteration by $|\Psi_k\rangle$ and θ_3^k we have:

$$
\begin{aligned}
|\Psi_1\rangle &= |\theta_1^1\rangle\,|0\rangle\,|\theta_1^1; \psi_3^{s_1,s_2}\rangle \Rightarrow \\
|\langle\varphi_3\,|\Psi\rangle|^2 &= \cos^2\theta_1^1\,|\langle\varphi_3\,|\psi_3^{s_1}\rangle|^2 + \sin^2\theta_1^1\,|\langle\varphi_3\,|\psi_3^{s_2}\rangle|^2;
\end{aligned}
$$

$$
\begin{aligned}
|\Psi_2\rangle &= |\theta_1^2\rangle\,|\theta_1^1\rangle\,|\theta_1^1 + (\theta_1^2 - \theta_1^1); \psi_3^{s_1,s_2}\rangle = |\theta_1^2\rangle\,|\theta_1^1\rangle\,|\theta_1^2; \psi_3^{s_1,s_2}\rangle \Rightarrow \\
|\langle\varphi_3\,|\Psi_2\rangle|^2 &= \cos^2\theta_1^2\,|\langle\varphi_3\,|\psi_3^{s_1}\rangle|^2 + \sin^2\theta_1^2\,|\langle\varphi_3\,|\psi_3^{s_2}\rangle|^2;
\end{aligned}
$$

$$
\vdots
$$

$$
\begin{aligned}
|\Psi_n\rangle &= |\theta_1^n\rangle\,|\theta_1^{n-1}\rangle\,|\theta_1^n; \psi_3^{s_1,s_2}\rangle \Rightarrow \\
|\langle\varphi_3\,|\Psi\rangle|^2 &= \cos^2\theta_1^n\,|\langle\varphi_3\,|\psi_3^{s_1}\rangle|^2 + \sin^2\theta_1^n\,|\langle\varphi_3\,|\psi_3^{s_2}\rangle|^2.
\end{aligned}
$$

A schematic showing the evolution of several representative consecutive fused observational error profiles is shown in Figure 4.2-2.

Before closing this section we note that if the rate of change $\dot{\theta}$ of the observational elevation is known, then the time evolution operator can be used to specify the observational error profile

state. In this case, the state is time dependent and at any time t is given by

$$\begin{aligned}
|\Psi(t)\rangle &= \hat{U}(t,t_0;\dot{\theta})\,|\theta_0;\psi^{s_1,s_2}\rangle \\
&= e^{-i\hat{\sigma}_3 \int_{t_0}^{t} \dot{\theta}dt}\,|\theta_0;\psi^{s_1,s_2}\rangle \\
&= \left|\theta_0 + \int_{t_0}^{t} \dot{\theta}dt;\psi^{s_1,s_2}\right\rangle,
\end{aligned}$$

where θ_0 is the elevation of the first observation made at time t_0. If $\dot{\theta}$ is constant, then

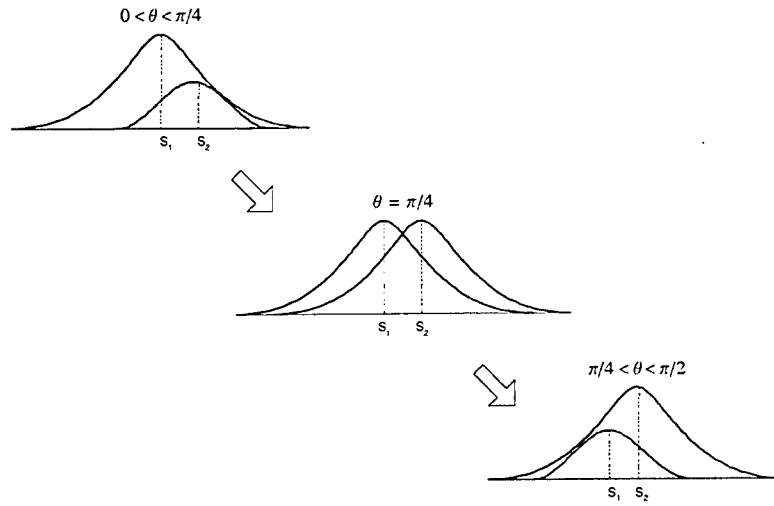$$|\Psi(t)\rangle = \left|\theta_0 + \dot{\theta}(t-t_0);\psi^{s_1,s_2}\right\rangle.$$

Figure 4.2-2. A Schematic Of The Evolution Of A Fused Observation Error Profile

5   OPTICAL COMPONENT DEVICE IMPLEMENTATIONS

In this section we will briefly provide "thumbnail sketches" of simple optical devices which:

1. Produce ε states for a logical variable;

2. Rotate child variable truth states; and

3. Measure the expected truth value and probability distribution for a logical variable's ε state.

It is not our intention to suggest that these simple devices be used for Dirac network application. Rather, we merely wish to illustrate that useful optical implementations of Dirac networks are possible. Practical general-purpose implementations would use state-of-the-art optical components - along with user/computer control capabilities which would provide the necessary flexibility to reconfigure the network in order to adapt it to a wide variety of application domains.

It should be pointed out that even though the theory developed above is based upon quantum mechanics, it is also valid within our context when classically intense (laser) light is used. Consequently, the properties which we discuss for each of these devices are valid at both the quantum level (i.e., for an individual photon) and the classical level (i.e. for large ensembles of identically prepared photons). In what follows, we shall assume that our light source is a laser which produces a classically intense, randomly polarized beam which exhibits a Gaussian flux density cross section; and each device is oriented within the same laboratory reference frame (as defined on accompanying figures).

5.1   OPTICAL ε STATES FOR LOGICAL VARIABLES

An optical ε state for a logical variable j can be easily produced using a laser, a polarization filter, and a specially cut bi-refringent prism (the prism has one facet cut at a small angle). The associated expected truth value and probability distribution can be measured using a slit screen and a CCD photon detector. This apparatus and its arrangement relative to the laboratory reference frame is shown in Figure 5.1-1., where: (i) the laser beam's center is aligned to lie precisely along the z-axis of the frame; (ii) the facets of the prism are oriented as shown in the figure with the facet nearest the laser normal to the beam; (iii) the beam covers a narrow slit oriented along the y-axis in the screen; and (iv) the CCD detector is centered in the beam and is oriented to sample the light intensity along the x-axis. The linear polarization state of a beam photon produced by the polarization filter defines the truth state for variable j. The natural Gaussian flux density profile of the beam provides a spatial probability distribution for finding a photon along the x-axis of the beam. The prism serves as an optical Stern-Gerlach apparatus and entangles the polarization states with polarization dependent probability amplitudes. The slit in the screen serves to disperse photons along the x-axis in order to produce a more easily observed probability distribution.

After passing through the polarization filter, the π state for variable j is

$$|\Phi\rangle = |\theta_j\rangle \left|\psi_j\right\rangle,$$

where $\left|\psi_j\right\rangle$ is the probability amplitude for finding a photon along the x-axis and $|\theta_j\rangle$ is the polarization state for a photon in the laboratory reference frame given by

$$|\theta_j\rangle = \cos\theta_j\,|x\rangle + \sin\theta_j\,|y\rangle.$$

Here, $|x\rangle$ and $|y\rangle$ are unit vectors along the system's x- and y-axes and represent the truth kets $|+\rangle_j$ and $|-\rangle_j$, respectively. The angle θ_j is the angular setting of the polarization filter measured clockwise (when looking along the z-axis from the laser to the detector) from the positive x-axis. After passing through the prism, a photon is produced which is in the polarization-position amplitude entangled ε state

$$\left|\theta_j;\psi_j^{x,y}\right\rangle = \cos\theta_j\,|x\rangle\left|\psi_j^x\right\rangle + \sin\theta_j\,|y\rangle\left|\psi_j^y\right\rangle.$$
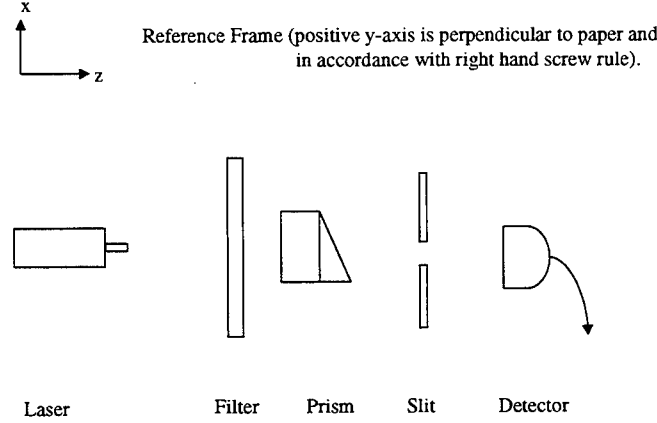
The photon intensity measurement by the CCD detector produces the probability distribution profile (along the x-axis) given by

$$\left|\left\langle x\left|\theta_j;\psi_j^{x,y}\right\rangle\right|^2 = \cos^2\theta_j\left|\left\langle x\left|\psi_j^x\right\rangle\right|^2 + \sin^2\theta_j\left|\left\langle x\left|\psi_j^y\right\rangle\right|^2.\right.\right.\right.$$

The system is calibrated by finding the CCD pixel numbers n_{θ_j} that locate the peaks of the $\theta_j = 0$, $\theta_j = \frac{\pi}{4}$, and $\theta_j = \frac{\pi}{2}$ intensity profiles. These correspond to the pixel numbers for expected truth values $+1$, 0, and -1, respectively (it is assumed that pixel numbers are counting numbers and they increase in the positive x direction such that $n_0 > n_{\frac{\pi}{4}} > n_{\frac{\pi}{2}}$). If n_α is the pixel number for the peak intensity for an arbitrary filter setting $0 \le \theta_j = \alpha \le \frac{\pi}{2}$, then the expected truth value for the variable can be calculated using

$$\langle\hat{\tau}_j\rangle = \frac{2\left(n_\alpha - n_{\frac{\pi}{4}}\right)}{n_0 - n_{\frac{\pi}{2}}}.$$

A $\left|\theta_j'\right\rangle$ post-selected ε state can be produced using the same apparatus as shown in Figure 5.1-1 with an additional post-selection polarization filter with angular setting θ_j' inserted in the beam between the prism and the slit. The resulting photon intensity profile is given by Equation(3-9) with φ_j replaced by x.

x

Reference Frame (positive y-axis is perpendicular to paper and
in accordance with right hand screw rule).

z

Laser   Filter   Prism   Slit   Detector

Figure 5.1-1. An Apparatus For Creating And Measuring An ε State

5.2 TIME-DEPENDENT OPTICAL INFLUENCES OF A PARENT UPON ITS CHILDREN

Consider an n variable system in which the time dependent variable 1 is the only parent for child variables $2, 3, \cdots, n$ and assume that simultaneous time dependent post-state expected truth value profiles $\langle \hat{\tau}_j(t) \rangle$, $t \geq t_0$, are required for variables $j = 2, 3, \cdots, n$. The time-dependent truth state for such a system is given by the specification

$$|\Psi(t)\rangle = \hat{U}_1(t, t_0; \dot{\theta}_1) \hat{R}_{12}(\alpha_{12}) \hat{R}_{13}(\alpha_{13}) \cdots \hat{R}_{1n}(\alpha_{1n}) |\theta_1\rangle |\theta_2\rangle \cdots |\theta_n\rangle ,$$

where - for the sake of simplicity - we will assume that $\alpha_{1j} = -\theta_1 \mp \gamma_{1j}$, with γ_{1j} a constant.

An optical realization of this system can be achieved using a classically intense randomly polarized laser as a photon source; a polarization filter with computer (programmed with the dynamics defined by $\dot{\theta}_1$) controlled time stepped setting θ_1 which defines $\hat{U}_1(t, t_0; \dot{\theta}_1) |\theta_1\rangle$; a 1-input port/$(n-1)$-output port polarization preserving beam splitter; $n-1$ optically active light guides tuned to perform the specified $n-1$ rotations $\hat{R}_{1j}(\alpha_{1j})$; and $n-1$ optically active light guides which induce the specified prior states $|\theta_j\rangle$, $j = 2, 3, \cdots, n$ into $|\Psi(t)\rangle$. Measurement of the required expected truth value time profile $\langle \hat{\tau}_j(t) \rangle$ for the jth child variable is performed using a computer-controlled rotation angle stepped polarization filter which operates in concert with a photon-intensity counter in order to rapidly determine and record the child's time-dependent filter setting $\theta'_j(t)$ which produces the maximum intensity at time t. This setting corresponds to the time-dependent post-state $\left| \theta'_j(t) \right\rangle$ which is used to compute and record the time-dependent expected truth value profile according to

$$\langle \hat{\tau}_j(t) \rangle = \cos^2 \theta'_j(t) - \sin^2 \theta'_j(t).$$

Note that this measurement apparatus seeks a filter setting θ_j'' such that the autocorrelation function $\langle \theta_j'' | \theta_j' \rangle = 1$, i.e. $\theta_j'' = \theta_j'$. A schematic of this apparatus is shown in Figure 5.1-2.

Let us examine in more detail how the optical activity of the light guides produce a rotation of the child prior state $|\theta_j\rangle$. The optically active medium of the light guide connecting variable 1 (input port) with that of variable j is characterized by a *specific rotatory power* δ_{1j} which is the measure of the change of the input truth state polarization angle θ_1 per unit length. Consequently, the associated rotation angle β_{1j} produced by a light guide of length ℓ_{1j} depends upon θ_1 and is given by the linear expression

$$\beta_{1j} = -\alpha_{1j} = \theta_1 \pm \delta_{1j}\ell_{1j},$$

where $+$ $(-)$ applies when the optically active medium is *dextrorotatory (levorotatory)*. If δ_j is the specific rotatory power for the optically active medium used to produce the prior child state $|\theta_j\rangle$, then the length ℓ_j of dextrorotatory medium required to rotate an input polarization vector through a positive angle θ_j is

$$\ell_j = \frac{\theta_j}{\delta_j},$$

so that the polarization vector entering the $|\theta_j\rangle$ child state medium at angle β_{1j} leaves it after traversing length ℓ_j at angle $\theta_j + \beta_{1,j}$. Therefore,

$$\begin{aligned} \hat{R}_{1j}(\alpha_{1j})|\theta_j\rangle &= |\theta_j - \alpha_{1j}\rangle \\ &= |\theta_j + \theta_1 \pm \delta_{1j}\ell_{1j}\rangle \\ &= |\theta_1 + \delta_j\ell_j \pm \delta_{1j}\ell_{1j}\rangle. \end{aligned}$$

As an illustration of this, assume that the laser produces 6500 wavelength randomly polarized light and that the light guide medium is quatrz which has a specific rotatory power $\delta_j = \delta_{1j} = 17^0/mm$ at 6500 wavelength. Suppose that the prior truth state for child variable j is $|34_j^0\rangle$. Then the length of the levorotatory light guide medium needed to produce this prior truth state for variable j is

$$\ell_j = \frac{34^0}{17^0/mm} = 2.0 \ mm.$$

If the required influence α_{1j} of variable 1 upon variable j is

$$-\alpha_{1j} = \beta_{1j} = \theta_1 - 8.5^0,$$

then the length of levrorotatory quartz light guide needed to implement this linear relationship is

$$\ell_{1j} = \frac{8.5^0}{17^0/mm} = 0.5 \ mm.$$

Consequently, post-state $|\theta_j'\rangle$ of variable j is given by

$$|\theta_j'\rangle = |\theta_j - \alpha_{1j}\rangle$$

$$= \left| \left(\theta_1 + 34^0 - 8.5^0 \right)_j \right\rangle$$

$$= \left| \left(\theta_1 + 25.5^0 \right)_j \right\rangle.$$

For example, if $|\theta_1\rangle = |10_1^0\rangle$, then $\left| \theta_j' \right\rangle = \left| 35.5_j^0 \right\rangle$. This example is shown in Figure $5.2 - 2$ where we follow the linear polarization (electric field) vector $\vec{E}$ of the light as it traverses the system. There the parent state $|\theta_1\rangle = |10_1^0\rangle$ is produced by a linear polarizer with angle setting $\theta_1 = 10^0$ relative to the laboratory reference frame's x-axis. Light leaves the polarizer and enters the light guide with $\vec{E}$ such that $\theta_1 = \arccos\left\{ \frac{\vec{E}}{|\vec{E}|} \cdot \frac{\vec{y}}{|\vec{y}|} \right\} = 10^0$. As the light traverses the $1j$ light guide, $\vec{E}$ is rotated counter-clockwise (looking from the polarizer down the light guide) and emerges after 0.5 mm with $\vec{E}$ making an angle 1.5^0 with the laboratory reference frame's positive x-axis. It then enters variable j's dextrorotatory medium with this $\vec{E}$ orientation and $\vec{E}$ is rotated clockwise as it traverses this light guide. The light leaves variable j's light guide - having been rotated 34^0 (clockwise) - with $\vec{E}$ making a final angle of $\theta_j' = 35.5^0$ with the x-axis.

Before closing, we note that a similar approach can be employed for the case where the roles of parent and child in Figure 9 are interchanged and there are $n - 1$ parents (variables 2 thru n) influencing a single child (variable 1). In this case, however, extra care must be exercised since the polarization states which exit the $j1$ light guides must be (vectorially) added along with that of the prior child state.
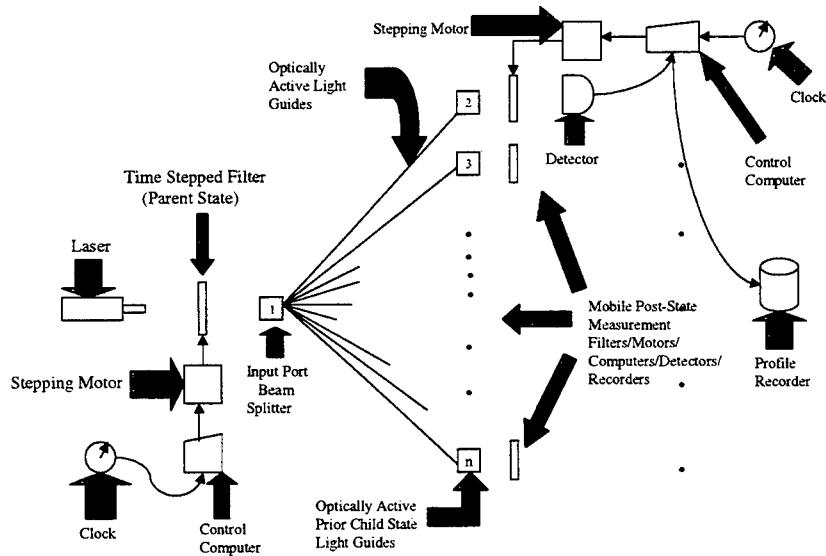

Figure 5.2-1. An Apparatus For Creating And Measuring The Influence Of A Time Dependent Parent Variable Upon Its Children
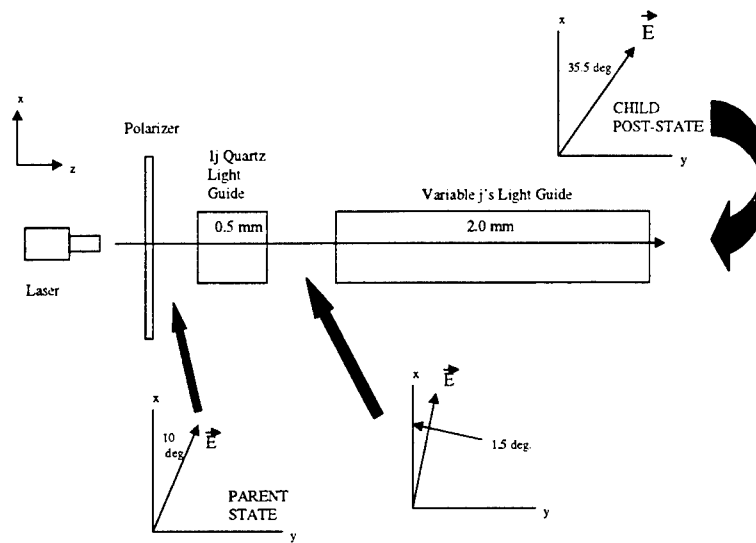
Figure 5.2-2. An Example Of Polarization State Rotations

6 CONCLUDING REMARKS

This report describes the research and development products generated during FY00 - FY01 that involve application of the Dirac algebra of quantum mechanics to methods of probabilistic inference. This work is viewed by the authors of this report as a *successful first step* towards the creation of a viable and extremely useful approach for automatic probabilistic reasoning, aspects of systems analysis, and data representation and analysis. Additional related areas that merit further investigation include the following:

- the use of complex valued probability amplitudes;

- using quantum interference to increase the probabilities for "good answers" and decrease the probabilities for "bad answers;"

- using general angular momentum states for multi-valued logic problems;

- the development of a user-friendly interface for a DIRACNET system;

- the development of inter-system interfaces for DIRACNET in order to enable informational updates for special application areas;

- the development of a complete optical implementation theory; and

- using DIRACNET to solve additional and more complex applications (e.g., data mining, data correlation and fusion, and temporal reasoning).

7 BIBLIOGRAPHY

[1] M. Chester, *Primer of Quantum Mechanics*, John Wiley and Sons, New York (1987).

[2] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics*, Volume 1, John Wiley and Sons, New York (1977).

[3] G. F. Cooper, "The Computational Complexity of Probabilistic Inference Using Bayesian Belief Networks", *Artificial Intelligence*, Vol. 42, p. 393 (1990).

[4] P. Dagum and M. Luby, "Approximating Probabilistic Inference in Bayesian Belief Networks is NP-hard", *Artificial Intelligence*, Vol. 60, p. 141 (1993).

[5] D. C. Fisher, "The Number of Words of Length n in a Graph Monoid", *The American Mathematical Monthly*, Aug-Sept 1989, p. 610.

[6] F. V. Jensen, *An Introduction to Bayesian Networks*, Springer-Verlag, New York (1996).

[7] R. E. Neapolitan, *Probabilistic Reasoning in Expert Systems: Theory and Algorithms*, John Wiley and Sons, New York (1990).

[8] J. Pearl, *Causality: Models, Reasoning, and Inference*, Cambridge University Press, Cambridge (2000).

[9] J. A. Rosen and W. L. Smith, "Influence Net Modeling with Causal Strength: An Evolutionary Approach", *Command and Control Research and Technology Symposium* (1996).

[10] G. Shaefer, *The Art of Causal Conjecture*, The MIT Press, Cambridge, (1996).

[11] P. W. Shor, "Algorithms for Quantum Computation: Discrete Logarithms and Factoring", *Proceedings from the 35th Symposium on Foundations of Computer Science*, 1994, p. 124.

Appendix A

AN OUTLINE OF CODE FOR THE EXAMPLES

The purpose of the code was to test the behavior of Dirac nets on a group of problems that were simple, but large enough to be too tedious to do by hand. Some assumptions were made in the interest of short run times. One was to avoid calculating sines and cosines repeatedly. The angle of a truth state vector, θ, and the angle representing a rotation, α, were restricted to integer values and the sines and cosines for all integer angles beteen $0°$ and $90°$ were stored in a table. Code was written in ANSI C. Several versions of the program exist, usually to acommodate different methods for deriving the α's. The code fragments here are meant to display the methods used without the burden of reading declarations, allocations and input/output processing not relevant to the method being discussed.

A list structure and a pointer to it were used to store the rotation angle information:

```
typedef struct list *List_ptr;
struct list
{   int influencer;
    int alpha;
    List_ptr next;
} List;
```

The prior and posterior values of θ were stored in:
```
    int *theta, *theta_post;
```

The sines and cosines of angles from $0°$ to $90°$ were stored in:
```
    double *sc;
```
Note that the first half of the table held sines and the second half held cosines.

First, information relevant to a node was read in. This included its prior θ, an indicator of how the α values were to be computed or modified, the influencing nodes and the initial α's (if appropriate). The α's were computed in a variety of ways, and some of the alternatives will be reviewed below. Once the α's were computed, the posterior θ's were calculated in a routine called update_angles:
```
void update_angles (int *theta, int *theta_post, int node, List_ptr *head_infl_list)
{   int angle;
    List_ptr x;
    int j;

    j = node;
    {   angle = *(theta + j);
        x = *(head_infl_list + j);
        while (x != NIL)
        {   angle = angle - x->alpha;
            x = x->next;
        }
        if (angle > 90) angle = 90;
        if (angle < 0) angle = 0;
        *(theta_post + j) = angle;
```

```
        }
}
```

As the code indicates, each influencing parent node simply contributes an α in an additive fashion to the total influence on the child node, subject to the doorstop rule that contrains θ to lie between 0° and 90°.

The expected truth values for both the prior and posterior values of the θ's are computed using the prior_truth and post_truth routines:

```
void prior_truth (int no_nodes, int *theta, double *sc, int no_angles, FILE *outfile)
{
    double expec_truth;
    double s, c;        /* Note: sin is at i and cos is at i+no_angles in sc    */
    int i;
    int t;

    for (i = 0; i < no_nodes; i++)
    {   t = *(theta + i);
        s = *(sc + t);
        c = *(sc + t + no_angles);
        expec_truth = c*c - s*s;
        fprintf (outfile," theta[%2d] = %2d   prior truth val = %f\n",
                i+1,t,expec_truth);
    }
}


void post_truth (int no_nodes, int *theta_post, double *sc, int no_angles, FILE *outfile)
{
    double expec_truth;
    double s, c;        /* Note: sin is at i and cos is at i+no_angles in sc    */
    int i;
    int t;

    fprintf (outfile,"\n");
    for (i = 0; i < no_nodes; i++)
    {   t = *(theta_post + i);
        s = *(sc + t);
        c = *(sc + t + no_angles);
        expec_truth = c*c - s*s;
        fprintf (outfile," theta[%2d] = %2d   post truth val = %f\n",
                i+1,t,expec_truth);
    }
}
```

Both these routines write their results to a formatted output file for the analyst.

Since it is possible to examine different combinations of nodes to get measures of interest, some method of tracking the nodes of interest and the corresponding states (true or false) of interest was required. The method chosen used two words, an integer and an unsigned integer, to do the tracking. If we use a convention where we label each bit from 1 to 32, starting at the right or low-end bit, then each bit can correspond to a node. (For the experiments we were running, 32 nodes were adequate, but if additional nodes were needed the scheme could be extended to multiple words.) The variable maskin is used to track which variables are of interest and the variable interest tracks whether the

true or false state of the particular variable is of interest. The user provides a list of variables, each followed by T or F, to indicate the variables and states of interest. The following code sets up the appropriate masks:

```
/* Read the number of nodes whose state is a state of interset          */
    fscanf (infile,"%d",&no_of_interest);
    node_of_interest = (int *) calloc(no_of_interest,sizeof(int));

/* Based on the number of nodes and nodes of interest, set up a mask for use in ket_prod  */
/*  Note that a node of interest can be of interest when it is in either the T of F state.  */
/*  This is indicated by the character (T or F) followoing the node number.          */
    mask = (unsigned int *) calloc(no_nodes, sizeof(unsigned int));
    for (i = 0; i < no_nodes; i++)
        *(mask + i) = pow(2,i);

    fprintf (outfile,"\n The nodes of interest and their state of interest are:\n");
    interest = 0;   maskin = 0;
    for (i = 0; i < no_of_interest; i++)
    { fscanf (infile,"%d %1s", &node, &char_tf);
        fprintf (outfile,"    %d  %c\n", node, char_tf[0]);
        *(node_of_interest + i) = node;
        temp = pow(2,(node - 1));
        maskin = maskin | temp;
        if (char_tf[0] == 'T')
            interest = interest | temp;
    }
    no_states = pow((double)2,(double)(no_nodes - no_of_interest));
    max_states = pow((double)2,(double)no_nodes);
```

Depending on the study being conducted, some of this code could be eliminated or simply hard-wired, if that proves more convenient.

Once the nodes and states of interest are established, the ket_prod routine is called. This computes the probability amplitude for the combination of nodes/states of interest. If the user is interested in those states where, for example, two nodes are both true, this routine (with the above initialization) can provide the result. See Section II C. Note that there is also a routine called all_prod which computes the probability amplitudes for all states. This can be of interest for small nets, but since the number of states goes as 2^N, the volume of output can rapidly become too large to provide much insight.

```
void ket_prod (int *theta_post, int no_nodes, int max_states, int interest, unsigned int *mask,
                double *prod_sum, int maskin, int no_angles, double *sc)
{
   int angle;
   int i, j;
   int offset;
   int temp, temp2;
   int state_val;
   double prod;

   *prod_sum = 0.;
   for (i = 0; i < max_states; i++)
   { temp = maskin & i;
     if (temp == interest)
     { state_val = i;
```

```
        prod = 1.;
        for (j = 0; j < no_nodes; j++)
        { angle = *(theta_post + j);
          temp2 = *(mask + j) & state_val;
          if (temp2 == 0)
           offset = angle;
          else
           offset = angle + no_angles;
           prod = prod * (*(sc + offset));
        }
        *prod_sum = *prod_sum + prod*prod;
    }
  }
)
```

Since the von Neumann entropy is available (see Section II H) for both individual nodes and groups of nodes, these calculations are performed in entropy_calc1 and entropy_calc2. The entropy ratio (the uncertainty associated with the group of nodes of interest relative to the entropy of the whole system) is the measure of interest, rater than the entropy itself, so the ratio is the result returned. The code for node groups follows. A similar routine does single nodes.

```
void entropy_calc2 (double *entropy, int *node_of_interest, int no_of_interest,
                    double *entropy_ratio)
{ int i;
  int count, loc;
  double sum;

  sum = 0.;
  count = 0;
  for (i = 0; i < no_of_interest; i++)
  { loc = *(node_of_interest + i);
    sum = sum + *(entropy + loc - 1);
    count++;
  }
  *(entropy_ratio) = sum/(count*LN2);
```

Several different approaches were used to set up the rotation angles, α. One was to simply read in an estimated α and use it directly. In the probability models, there was a two-way conversion between probabilities and angular settings. The code for this version of program is:

```
void func6 (List_ptr *head_infl_list, int *save, int node, int icount, int* theta,
            int *theta_post)
{ int gi, hi;
  double gf, hf;
  List_ptr x, y;
  int j;

/* If the form of influence is 6, a list consisting of one or more integers follows.  The    */
/* integers are the influencing nodes & the + and - influencing factors, entered as integers.*/
/* The overall node ordering is important since it is the updated angles that influence later*/
/* nodes.                                                                                     */

  x = lalloc();
  *(head_infl_list + node) = x;
  x->influencer = *(save+1);            /* influencing node          */
  gi = *(save + 2);
```

```
  hi = *(save + 3);
  gf = (float)gi/100.;
  hf = (float)hi/100.;
  gf = (float)gi/100.;
  x->g = gf;
  x->h = hf;
  x->next = NIL;
  j = 4;

  while (j < icount)
  { y = lalloc();
    y->influencer = *(save + j);
    gi = *(save + j + 1);
    hi = *(save + j + 2);
    gf = (float)gi/100.;
    hf = (float)hi/100.;
    y->g = gf;
    y->h = hf;
    y->next = NIL;
    x->next = y;
    x = y;
    j = j + 3;
  }
}

/*--------------------------------------------------------------------------------*/

void update_angles (int *theta, int *theta_post, int node, List_ptr *head_infl_list)
{ int angle;
  int a2, infl, th;
  double thfl, a2fl, anglefl;
  List_ptr x;
  int j;

  j = node;
  th = *(theta + j);
  *(theta_post + j) = th;
  thfl = (float)th;
  x = *(head_infl_list + j);
  while (x != NIL)
  { infl = x->influencer;
    a2 = *(theta_post + infl - 1);
    a2fl = (float)a2;
    anglefl = ((90. - a2fl)/90.) * ((90. - thfl)/90. + (thfl/90.)*x->g)
            + (a2fl/90.) * ((90. - thfl)/90. + (thfl/90.)*x->h);
    angle = (int)(90. - 90.*anglefl);
    x = x->next;
    thfl = 90. - 90.*anglefl;
  }
  if (angle > 90) angle = 90;
  if (angle < 0) angle = 0;
  *(theta_post + j) = angle;
}
```

This code implements the process described in Section IV C, Case 1.

For some of the problems a sigmoid function seemed the most appropriate, so that if a parent is uncertain or nearly so, it influences the child minimally, and if it is very likely to be true it strongly

affects the child in a postive (or negative) fashion, and if it is very likely to be false it affects the child in the opposite direction. The code to implement this process is the following:

```
void func2 (List_ptr *head_infl_list, int *save, int node, int icount, int* theta,
            int *theta_post)
{ int temp, temp2, beta;
  double scale, betafloat;
  List_ptr x, y;
  int j;

/* If the form of influence is 2, a list consisting of a single beta, followed by one or more*/
/* pairs of integers.  Beta is entered as an integer and interpreted as a percent.  An entry*/
/* of 50 yields a factor of 0.5.  The first of each pair is the influencing (parent) node.  */
/* The second of each pair is the angle of influence, alpha.  A sigmoid funtion is computed */
/* based on these values.  The entered alpha is modified: alpha = orig.alpha * scale.       */
/*    scale = 2*( (1./(1. + e**(-beta*(45 - theta_parent))) - 0.5)                          */
/* Scale is thus a function of the form 1./(1.+e**x), adjusted to range from -1. to 1.      */

  x = lalloc();
  *(head_infl_list + node) = x;
  beta = *(save+1);
  betafloat = ((double)beta)/100.;
  x->influencer = *(save+2);                /* influencing node        */
  temp2 = *(theta_post + x->influencer - 1);
  temp = *(save+3);                         /* alpha before scaling    */
  scale = 1./(1.+ exp(-betafloat*(double)(45 - temp2)));
  scale = 2.*(scale - 0.5);
  x->alpha = (int)(scale*temp);
  x->next = NIL;
  j = 4;
  while (j < icount)
  { y = lalloc();
    y->influencer = *(save + j);
    temp2 = *(theta_post + y->influencer - 1);
    temp = *(save + j + 1);
    scale = 1./(1.+ exp(-betafloat*(double)(45 - temp2)));
    scale = 2.*(scale - 0.5);
    y->alpha = (int)(scale*temp);
    y->next = NIL;
    x->next = y;
    x = y;
    j = j + 2;
  }
}
```

Because many versions of the program were developed and one version would usually cannibalize another, with appropriate "commenting out", a complete listing is not given. For those willing to hack through experimental code in C, an operational version can be provided as a text file. Send requests to ParksAD@nswc.navy.mil.

DISTRIBUTION

DOD ACTIVITIES (CONUS)

	Copies
PROGRAM EXECUTIVE OFFICE FOR SURFACE COMBATANTS/ AEGIS PROGRAM ATTN CDR JAMES C TERTOCHA PMS400B3A 2531 JEFFERSON DAVIS HWY ARLINGTON VA 22242-5165	1
DEFENSE TECHNICAL INFORMATION CENTER 8725 JOHN J KINGMAN ROAD FT BELVOIR VA 22060-6218	1
ATTN CODE A76 (TECHNICAL LIBRARY) COMMANDING OFFICER CSSDD NSWC 6703 WEST HWY 98 PANAMA CITY FL 32407-7001	1

NON-DOD ACTIVITIES (CONUS)

	Copies
ATTN DR RABINDER N MADAN OFFICE OF NAVAL RESEARCH CODE 313 800 NORTH QUINCY STREET ARLINGTON VA 22217	1
ATTN RANDY MORRIS 137-227 LOCKHEED MARTIN GES 199 BORTON LANDING ROAD MOORESTOWN NJ 08057	1

	Copies
ATTN DR FRANK REIFLER 137-227 LOCKHEED MARTIN GES 199 BORTON LANDING ROAD MOORESTOWN NJ 08057	1
ATTN DR STEPHEN R ADDISON DEPARTMENT OF PHYSICS AND ASTRONOMY UNIVERSITY OF CENTRAL ARKANSAS CONWAY AR 72035	1

INTERNAL

		Copies
B05	TRIOLA	1
B05	TATE	1
B10	PARKS	3
B30	LEITE	1
B30	BALCHIN	1
B30	SPENCE	1
B32	GRAY	1
B32	MCCABE	1
B32	SMITH-CARROLL	1
B32	KUCHINSKI	1
B60	TECHNICAL LIBRARY	3
B60	GRAY	1
L04	HARMAN	1
T12	HERMANN	1